Optional Mathematics

Grade 9

Government of Nepal
Ministry of Education, Science and Technology
Curriculum Development Centre
Sanothimi, Bhaktapur

Publisher: Government of Nepal
Ministry of Education, Science and Technology
Curriculum Development Centre
Sanothimi, Bhakapur

First Edition: 2082 BS

PREFACE

School education is the foundation for preparing the citizen who are loyal to the nation and nationality, committed to the norms and values of federal democratic republic, self-reliant and respecting the social and cultural diversity. It is also remarkable for developing a good moral character with the practical know-how of the use of ICT along with the application of scientific concept and positive thinking. It is also expected to prepare the citizens who are moral and ethical, disciplined, social and human value sensitive with the consciousness about the environmental conversation and sustainable development. Moreover, it should be helpful for developing the skills for solving the real life problems. This textbook 'Optional Mathematics, Grade 9' is fully aligned with the intent carried out by the National Curriculum Framework for School Education, 2076 and is developed fully in accordance with the new Secondary Level Optional Mathematics Curriculum (Grade 9-10), 2081.

This textbook was developed by Mr. Shakti Prasad Acharya, Mr. Sujan Kafle and Mr. Anil Kumar Jha. It was translated by Mr. Sushil Khanal, Mr. Ramhari Shrestha, Mr. Surath Khadka, Ms. Maiya Khadka and Mr. Jagannath Adhikari. The contribution made by Director General Mr. Yubaraj Paudel, Mr. Ima Narayan Shrestha, Prof. Dr. Hari Prasad Upadhyaya, Ms. Pramila Bakhati, Mr. Gyanendra Ban, Ms. Anupama Sharma, Mr. Navin Poudel, Mr. Satya Narayan Maharjan, Dr. Shyam Prasad Acharya, Dr. Tikaram Pokharel, Mr. Ramchandra Dhakal, Mr. Rajukanta Acharya, Mr. Saroj Bhakta Acharya, Ms. Indira Pandey, Mr. Binod Thapaliya, Mr. Mahesh Bhattarai and Mr. Kamal Nepal is remarkable in bringing the book in this form. The language of this book was edited by Mr. Bijaya Kumar Ranabhat and the layout design of the book was done by Mr. Jayaram Kuikel. The Centre expresses its heartfelt gratitude to everyone involved in its development.

The textbook is a primary resource for classroom teaching. Considerable efforts have been made to make the book helpful in achieving the expected competencies of the curriculum. Curriculum Development Centre always welcomes constructive feedback for further betterment of its publications.

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Contents

Lesson 1

Algebra

1.1 Relation and Function

A mathematical statement that clearly defines the relationship between two variables is called a function. The term function was first used in 1673 by the German mathematician Leibniz to represent the dependency of one quantity to another.

Functions are found to be used in various activities of daily life as well. The relationship between time and amount of interest when investing a fixed sum at a fixed interest rate, the relationship between distance and time when an object moves at a uniform speed, are the examples of functions. Functions are used in various fields such as mathematics, science, computer science, economics, engineering, and medical science.

Leibniz

1.1.1 Ordered Pair

Complete the table below:

Set A	Set B	In pairs	In ordered pair
2	4	(4, 2) (2, 4)	(2, 4)
3	6	……..	
4	8	……..	
5	10	……..	

Discuss the following questions:

a. In the ordered pairs (2, 4), (3, 6), (4, 8), (5, 10), what is the relationship between the value of each element of set A and the corresponding element of set B?

b. If the order of the elements in the pair are changed, what effect occurs in the relationship between them?

c. Does the relation remain same? Explore it.

Here, in the ordered pairs (2, 4), (3, 6), (4, 8), (5, 10), the elements from A are the half of the elements from B. If the order of these elements is reversed, it becomes (4, 2), (6, 3), (8, 4), (10, 5). Now, the elements from set A become the double of the elements from set B. Therefore, when we write these elements by reversing their order, the relation also changes. The pair that will change value and meaning after changing the order of the elements is called an ordered pair.

Thought Provoking Question: Do the pairs (4, 5) and (5, 4) represent the same point? Draw the conclusion by representing them in a graph.

In the ordered pair, the order of the element is significant. When the order is changed, that relation is changed as well. In the ordered pair (4, 5), 4 is the first and 5 is the second element.

For two ordered pairs to be equal, their corresponding elements must be equal.

If $(a, b) = (c, d)$, then $a = c$ and $b = d$. For example, $(4, \frac{6}{2}) = (\frac{12}{3}, 3)$ If $(a, b) = (b, a)$, then $a = b$ must be true.

The pair of elements in a specific order that has a definite relationship, enclosed within small brackets (parenthesis) and separated by a comma (,), is called or ordered pair. The ordered pair is written in the form (x, y), where x is called the antecedent and y is called the consequent.

Example 1

If the ordered pairs $(x, 8)$ and $(4, y)$ are equal, find the values of x and y.

Solution

Here, $(x, 8) = (4, y)$

Equating the corresponding elements of the equal ordered pairs,

$\therefore x = 4$ and $y = 8$

Example 2

If the ordered pairs $(a, 3)$ and $(5, b)$ satisfy the relation $3x + 5y = 30$, find the values of the ordered pair (a, b).

Solution

The ordered pair $(a, 3)$ satisfies the relation $3x + 5y = 30$, So, $3 \times a + 5 \times 3 = 30$ Or, $3a + 15 = 30$ Or, $3a = 30 - 15$ Or, $3a = 15$ $\therefore a = 5$	The ordered pair $(5, b)$ satisfies the relation $3x + 5y = 30$, So, $3 \times 5 + 5 \times b = 30$ Or, $15 + 5b = 30$ Or, $5b = 30 - 15$ Or, $5b = 15$ $\therefore b = 3$

The required ordered pair, $(a, b) = (5, 3)$.

Exercise 1.1 (A)

1. a. Define ordered pair with an example.
   b. What kind of ordered pairs are equal order pairs? Write with an example.
   c. If $(a, b) = (m, n)$, state the relation between a and m.
2. Find the values of x and y in the following cases:
   a. $(x, 4) = (5, y)$ b. $(x - 1, y + 2) = (6, 7)$
   c. $(x - 3, y + 7) = (2, -5)$ d. $(2x - 5, 4) = (9, y + 4)$
   e. $(\frac{x}{3} + 1, y - \frac{2}{3}) = (\frac{5}{3}, \frac{2}{3})$ f. $(2x + y, 4) = (6, 3x - y)$
3. a. For what values of p and q will the ordered pairs $(2p + 4, p - q)$ and $(8, q)$ be equal? Find them.
   b. For what values of m and n will the ordered pairs $(3m - 5, 2n + m)$ and $(7, m + 2)$ be equal?
4. If the ordered pairs satisfy the relation $y = 2x + 4$, find the remaining elements of the ordered pairs:
   a. (1, …) b. (3, …) c. (…, 0) d. (…, –2)
5. a. If the ordered pairs $(3, p)$ and $(q, 4)$ satisfy the relation $4x + 3y = 24$, find the values of order pair (p, q).
   b. If the ordered pairs $(3, m)$ and $(n, 7)$ satisfy the relation $7x - 3y = 21$, find the values of m and n.

Practical Work

Collect the number of class-wise students from class 1 to 10 of your school and write the class and the number of students in ordered pairs.

Answer

1. Show to the teacher.

2. a. $x = 5, y = 4$ b. $x = 7, y = 5$ c. $x = 5, y = -12$ d. $x = 7, y = 0$
   e. $x = 2, y = \frac{4}{3}$ f. $x = 2, y = 2$

3. a. $p = 2, q = 1$ b. $m = 4, n = 1$

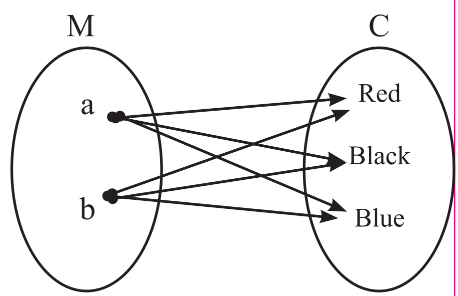
4. a. 6 b. 10 c. –2 d. –3

5. a. (4, 3) b. (0, 6)

1.1.2 Cartesian Product

Activity 1

The figure on the right shows the relationship between the models of motorcycles produced by a company, represented by set M and their colours, represented by set C. Make a set of all order pairs.



Procedure: Each student should solve the above problem in collaboration with the teacher and write their conclusion about the set thus formed.

Here, M = {*a*, *b*} and C = {red, black, blue}. Presenting the production model and colour of motorcycles of the company in ordered pairs,

M × C = {(*a*, red), (a, black), (*a*, blue), (*b*, red), (*b*, black), (*b*, blue)}. This set is called the cartesian product of sets M and C.

Thought Provoking Question: As above, write the ordered pairs for C × M. Are M × C and C × M said to be equal sets? Explore it.

If two sets A and B are non-empty set, then the set of all ordered pairs formed by taking the first element from the set A and the second element from set B is called the Cartesian product of A and B.

Thus, $A \times B = \{(x, y) : x \in A, x \in B\}$

Ways of Representing Cartesian Product

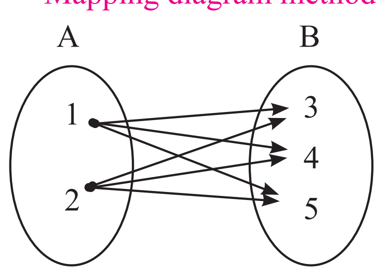
Let A= {1, 2} and B = {3, 4, 5}. The ways of representing A × B are as follows:

Cartesian product A × B = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)}

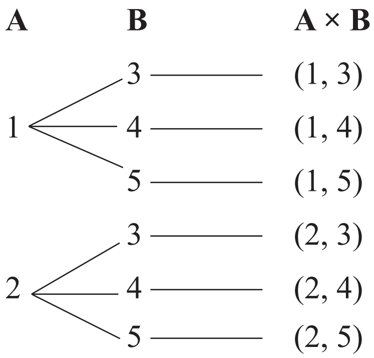
Tabulation method

	B			
A	×	3	4	5
	1	(1, 3)	(1, 4)	(1, 5)
	2	(2, 3)	(2, 4)	(2, 5)

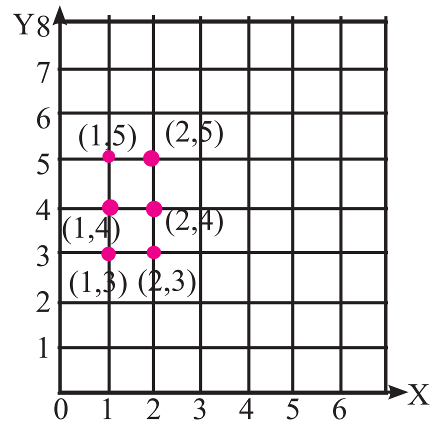
Mapping diagram method



Tree diagram method



Graphical method



Example 1

If A × B = {(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}, find the sets A and B. Also, find na., nb. and n(A × B). Are A × B and B × A equal? Write it.

Solution

Here, A × B = {(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}

A = Set of first element of each order pair ={2, 3}

B = Set of second element of each order pair = {4, 5, 6}

Therefore, na. = 2, nb. = 3 and n(A × B) = 6.

Now, B × A = {(4, 2), (5, 2), (6, 2), (4, 3), (5, 3), (6, 3)}

Therefore, A × B $\neq$ B × A.

Example 2

If A = {x: $x \leq$ 5, $x \in$ N} and B= {x: $x^2 - 4 = 0$, $x \in$ Z}, find A × B and B × A, where N represents set of natural numbers and Z represents set of integers.

Solution

A = { x: $x \leq 5$, $x \in$ N} = {1, 2, 3, 4, 5},

B = {x : $x^2 - 4 = 0$} = {–2, 2}

Now,

A × B = {(1, –2), (1, 2), (2, –2), (2, 2), (3, –2), (3, 2), (4, –2), (4, 2), (5, – 2), (5, 2)}

B × A = {(–2, 1) (–2, 2), (–2, 3), (–2, 4), (–2, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5)}

$x^2 - 4 = 0$

or, $(x - 2)(x + 2) = 0$

or, $x - 2 = 0, x + 2 = 0$

or, $x = 2, x = -2$

Exercise 1.1(B)

1. a. What is cartesian product? Write with an example.

   b. If the cardinal numbers of sets A and B are 3 and 2 respectively, what is the cardinal number of $A \times B$? Write it.

   c. If $n(A \times B) = x$, $n(A) = y$ and $n(A) = z$, write the relationship among x, y and z.

2. a. If $A = \{2, 3\}$ and $B = \{7\}$, find $A \times B$ and represent it in graph and mapping diagram.

   b. If $A = \{2, 3\}$ and $B = \{4, 5, 6\}$, find $B \times A$ and represent it in table and mapping diagram.

3. a. If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 1), (b, 5)\}$, find the values of set A, set B, $n(A)$, $n(B)$, $B \times A$ and $n(B \times A)$.

   b. If $A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4)\ (2, 5), (2, 6), (3, 4), (3, 5)\ (3, 6)\}$, find the values of set A, set B, $n(A)$, n (B), $B \times A$ and $n(B \times A)$.

4. a. If $A = \{x : x \leq 4, x \in N\}$ and $B = \{x : x^2 - 5x + 6 = 0, x \in N\}$, find $A \times B$ and $B \times A$. Also represent them in table, mapping diagram and tree diagram.

   b. If $M = \{x: x \leq 3, x \in N\}$ and $N = \{y : y = x - 1, x \in M\}$, find $M \times N$. Also, represent it in graph.

Answer

1. a. Show to the teacher. b. 6 c. $x = yz$

2. a. $A \times B = \{(2, 7), (3, 7)\}$, show the representation to the teacher.

   b. $B \times A = \{(4, 2), (4, 3), (5, 2), (5, 3), (6, 2), (6, 3)\}$, show the representation to the teacher.

3. a. $A = \{a, b\}$, $B = \{1, 2, 5\}$, $n(A) = 2$, $n(B) = 3$, $B \times A = \{(1, a), (1, b), (2, a), (2, b), (5, a), (5, b)\}$, $n(B \times A) = 6$, show the representation to the teacher

   b. $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $n(A) = 3$, $n(B) = 3$, $B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1)\ (5, 2), (5, 3), (6, 1), (6, 2)\ (6, 3)\}$, $n(B \times A) = 6$, show the representation to the teacher.

4. a. $A = \{1, 2, 3, 4\}$, $B = \{2, 3\}$, $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$, $B \times A = \{(2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$, show the representation to the teacher

   b. $M = \{1, 2, 3\}$, $N = \{0, 1, 2\}$, $M \times N = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$, show the representation to the teacher.

1.1.3 Relation

Activity 1

Study the relationship given below:

→ Shila is the daughter of Resham.

→ Sakina is the daughter of Milan.

→ Susmita is the daughter of Surendra.

Here, the relations are expressed in the ordered pair, it is (Shila, Resham), (Sakina, Milan) and (Susmita, Surendra). If the set of daughters is A = {Shila, Sakina, Susmita} and the set of fathers is B = {Resham, Milan, Surendra}, then the relation 'is the daughter of" exists between the elements of set A and set B. Assuming R for the relation, R = (Shila , Resham), (Sakina, Milan) and (Susmita, Surendra)}.

In the same way, the relation between elements of sets can be expressed as less than, greater than, equal to, double of, half of, square of, etc.

Here, what kind of relation do you find between R and A × B? Explore it.

Are all the elements of R in A × B? If so, can R said to be a subset of A × B?

The relation R on sets A and B is a subset of the Cartesian product of sets A and B.

Symbolically, it is written as $R \subseteq A \times B$.

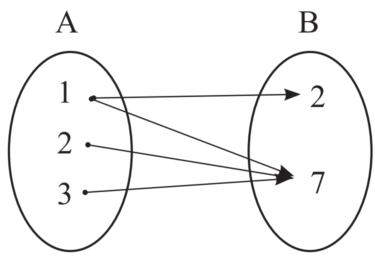
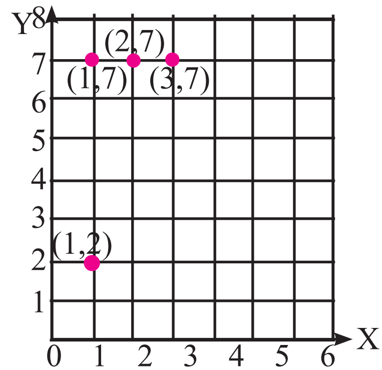
Thought Provoking Question: If one or both sets are empty, what type of relation can be defined? Discuss.

A subset of the Cartesian product of two sets is called a relation. If the relation between sets A and B is denoted by R, then $R \subseteq A \times B$. The subset R is defined according to the relation between the first and second elements of A × B. The second element of each ordered pair is called the image of the first element in the relation R.

Ways of Representing of a Relation

The relation R between any two sets A = {1, 2, 3} and B = {2, 7} is: "an element from set A is lesser than an element from set B."

Now, the Cartesian product can be written as: A × B = {(1, 2), (1, 7), (2, 2), (2, 7), (3, 2), (3, 7)} and the relation is: R = {(1, 2), (1, 7), (2, 7), (3, 7)}

<table>
<tr><th>In mapping diagram</th><th>In graph</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>In table form
<table>
<tr><td>x</td><td>1</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>y</td><td>2</td><td>7</td><td>7</td><td>7</td></tr>
</table></td><td>Set of ordered pairs
R = {(1, 2), (1, 7), (2, 7), (3, 7)}</td></tr>
<tr><td>In set builder form
R = {(x, y): x < y}</td><td></td></tr>
</table>

Example 1

If A × B = { (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)} then write ordered pair for the following relations:

a. is less than b. is equal to c. is square of d. is greater than e. is square root of

Solution

Here, A × B = { (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

a. "Is less than" relation $=\{(x, y) : x < y\}$ = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b. "Is equal to" relation: $\{(x, y) : x = y\}$ = {(2, 2), (3, 3)}

c. "Is square of" relation: $\{(x, y) : x = y^2\} = \phi$

d. "Is greater than" relation $\{(x, y) : x > y\}$ = {(3, 2)}

e. "Is square root of" relation: $\{ (x, y): x = \sqrt{y} \}$ = {(2, 4)}

Example 2

If a relation R = $\{(x, y): y = 2x , x < 4 \text{ and } x \in N\}$, find all the ordered pairs of the relation R.

Solution

Here, given relation R $= \{(x, y): y = 2x, x < 4, x \in N\}$, $x = 1, 2, 3$ and $y = 2x$

$x = 1, y = 2x = 2 \times 1 = 2$, so, ordered pair is (1, 2).

$x = 2, y = 2x = 2 \times 2 = 4$, so, ordered pair is (2, 4).

$x = 3, y = 2x = 2 \times 3 = 6$, so, ordered pair is (3, 6).

$\therefore$ R = {(1, 2), (2, 4) , (3, 6)}

Types of Relation

If A is a non-empty set and R is the subset of Cartesian product A × A, then R is the relation in A.

For example: If A = {1, 2, 3} and R = $\{(x, y): y = x^2\}$,

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}and R = {(1, 2)}

Here, the relation R is called a relation within the set A. There are mainly three kinds of relations in A.

A. Reflective Relation

Study the following examples:

a. The parallel relation between straight lines is reflexive because for any straight line *l*, *l* is parallel to *l*.

b. The congruence relation between geometric figures is reflexive because ΔABC ≅ ΔABC. Similarly, the relation of congruency among angles, line segments and polygons is also reflexive.

If every member of a set A is related to itself under a relation R, then such a relation R is called a reflexive relation.

For example, the equality relation among numbers is reflexive because for any number a, $a = a$.

If a relation R in *a* set A, for every element x of A is $(x, x) \in$ R or, R = $\{(x, x)$: for all $x \in A\}$, then such *a* relation R is called reflexive.

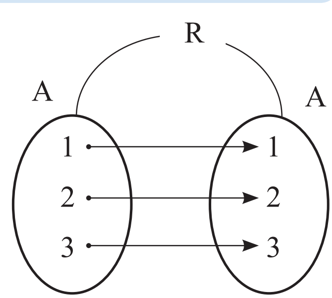
Example 1

If A = {1, 2, 3}, R = {(1, 1), (2, 2), (3, 3)}

$1 \in A \rightarrow (1, 1) \in R$

$2 \in A \rightarrow (2, 2) \in R$

$3 \in A \rightarrow (3, 3) \in R$

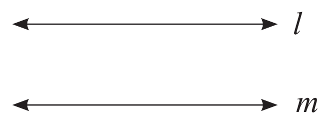


Here, every element of set A is related to itself, so, the relation R is *reflexive*.

Thought Provoking Question: Let $R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$, is R_1 said to be reflexive? Give reason.

B. Symmetric Relation

Study the following examples:



a. An equal relation between two numbers x and y is symmetric because $x = y$, then $y = x$.

b. A relation between two lines l and m is symmetric because, if $l//m$, then $m//l$.

c. A congruent relation between two geometric figures is symmetric. For example, if $\Delta ABC \cong \Delta DEF$ then $\Delta DEF \cong \Delta ABC$. Likewise, there is a congruency relations between angles, line segments and polygons, and this relation is symmetric.

If for any two elements x and y of a set, the relation of x to y also implies the relation of y to x, then such a relation is called symmetric.

A relation R in a set A is symmetric if for every $(x, y) \in R$ then $(y, x) \in R$ or $R = (x, y): (x, y) \in R \rightarrow (y, x) \in R, x, y \in A \}$

A relation $R: A \rightarrow A$ is symmetric relation if whenever xRy, then yRx also holds.

Example 2

If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 2), (3, 1)\}$ then

$(1, 1) \in R \rightarrow (1, 1) \in R$, $\quad (1, 2) \in R$ but $(2, 1) \notin R$

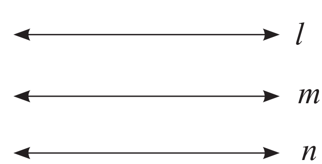
$(2, 2) \in R \rightarrow (2, 2) \in R$, $\quad (3, 1) \in R$ but $(1, 3) \notin R$

In above relation for any ordered pair, if $(x, y) \in R$, then $(y, x) \notin R$. The relation R is not symmetric.

Thought Provoking Question : Is $R_1 = \{(1, 2), (1, 1), (2, 1), (2, 3), (3, 2)\}$ said to be symmetric? Give reason.

C. Transitive Relation

Study the examples:



a. The equality relation among three numbers x, y and z is transitive because if $x = y$, $y = z$, then $x = z$.

b. The parallel relation among three straight lines l, m, and n is transitive because if $l \parallel m$ and $m \parallel n$ then $l \parallel n$.

c. The congruence relation among three geometric figures is also transitive.

For example, if $\Delta ABC \cong \Delta$ DEF and $\Delta DEF \cong \Delta GHI$, then $\Delta ABC \cong \Delta GHI$. Similarly, the relation of congruency among angles, line segments, and polygons are transitive.

In the same way, the relation between similar triangles is also a transitive relation.

In general, if in a set, an element x is related to another element y, and y has the same relation with another element z, and consequently x also has that same relation with z, then such a relation R is called a transitive relation.

A relation R on a set A is transitive if for any three elements $x, y, z \in R$ whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

In other words, for a relation R:A→A, if x R y and y R z, then x R z, then the relation R is transitive.

Thought Provoking Question: Is R a transitive if the set A={1, 2, 3} and the relation R={(1, 2), (2, 3), (1, 3)}? Give a reason.

Equivaence Relation

A relation is called an equivalence relation if it is reflexive, symmetric, and transitive. Relations such as equality, parallelism, and congruence are all equivalence relations.

Example 3

A = {1, 2, 3, 4} and a relation R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 4)}, is the relation R said to be an equivalence relation? Verify and write conclusion.

Solution

Given that, A = {1, 2, 3, 4} and R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 4)} is given. Let's check whether the relation R is reflexive, symmetric, and transitive or not.

a. Reflexive Relation

$1 \in A \rightarrow (1, 1) \in R$, $2 \in A \rightarrow (2, 2) \in R$

$3 \in A \rightarrow (3, 3) \in R$, $4 \in A \rightarrow (4, 4) \in R$

$\therefore$ For all elements x in set A, $(x, x) \in R$. Therefore, R is a reflexive relation.

b. Symmetric Relation

$(1, 1) \in R \rightarrow (1, 1) \in R$, $(2, 2) \in R \rightarrow (2, 2) \in R$

$(3, 3) \in R \rightarrow (3, 3) \in R$, $(4, 4) \in R \rightarrow (4, 4) \in R$

$(2, 3) \in R$, $(3, 2) \notin R$, $(3, 4) \in R$, $(4, 3) \notin R$

Since for all (x, y) in R, (y, x) is not in R and R is not symmetric.

Therefore, R cannot be said to be an equivalence relation.

Example 4

If M = {1, 2}, then find R = M × M, can R be called an equivalence relation? Verify it with conclusion.

Solution

Here, M ={1, 2}, R = M × M = {(1, 1), (1, 2), (2, 1), (2, 2)}

a. Reflexive

$1 \in M \rightarrow (1, 1) \in R$ $\quad$ $2 \in M \rightarrow (2, 2) \in R$

In this way, for all elements x in set M, $(x, x) \in R$. Therefore, R is called reflexive.

b. Symmetric

$(1, 1) \in R \rightarrow (1, 1) \in R$ $\quad$ $(1, 2) \in R \rightarrow (2, 1) \in R$

$(2, 1) \in R \rightarrow (1, 2) \in R$ $\quad$ $(2, 2) \in R \rightarrow (2, 2) \in R$

In this way, for all (*x*, y) in R (y, *x*) is also in R. Therefore, R is symmetric.

c. Transitive Relation

(1, 2) and $(2, 1) \in R \rightarrow (1, 1) \in R$ $\quad$ (1, 2) and $(2, 2) \in R \rightarrow (1, 2) \in R$

(2,1) and $(1, 1) \in R \rightarrow (2, 1) \in R$ $\quad$ (2, 1) and $(1, 2) \in R \rightarrow (2, 2) \in R$

∴ For all ordered pairs where (x, y) and (y, z) are in relation R (x, z) is also in relation R. Therefore, R is transitive.

Conclusion: From the above verification, the given relation R is reflexive, symmetric, and transitive. Therefore, R can be called an equivalence relation.

Exercise 1.1 (C)

1. a. Define the term 'Relation' with an example.
   b. What are the methods of representing a relation? Write it.
   c. Define the following relations with examples:
      i. Symmetric Relation $\quad$ ii. Reflexive Relation
      iii. Transitive Relation $\quad$ iv. Equivalence Relation
   d. If R = {(*a*, *a*), (*b*, *b*)}, what type of relation is R?
   e. If R = {(*a*, *b*)}, write the symmetric relation of R.
2. If A × B ={(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}, then find the relations given below and represent them in an arrow diagram:
   a. $R_1 = \{(x, y): x + y = 6\}$ $\quad$ b. $R_2 = \{(x, y): x < y\}$
   c. $R_3 = \{(x, y): y = x^2\}$

3. If A = {1, 2, 3} and B = {2, 3, 4}, find A × B and represent it in the form of a relation as given below.

a. is greater than b. is equal to c. is double of

d. is half of e. is square of

4. If A = {6, 7, 8, 10} and B = {2, 4, 6}, represent the relations A × B by following methods:

a. Set of ordered pairs b. Mapping diagram c. Graph d. Table

i. $R_1 = \{(x, y): x + y < 12, x \in A, x \in B\}$

ii. $R_2 = \{(x, y): 2x + y > 10, x \in A, y \in B\}$

5. The relation R is defined on the set A={1, 2, 3} where R ={ (1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}. Is R an equivalence relation? Verify it with calculation.

6. If A = {4, 5, 6}, find the followings:

a. A × A b. Reflexive relation R_1 in A

c. Symmetric relation R_2 in A d. Transitive relation R_3 in A

Answer

1. a. - c. Show to the teacher. d. Reflexive e. $R = \{(b, a)\}$

2. a. $R_1 = \{(1, 5), (2, 4)\}$ b. $R_2 = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

c. $R_3 = \{(2, 4)\}$ Show the mapping diagram to the teacher.

3. A × B = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

a. $R = \{(x, y): x > y\} = \{(3, 2)\}$ b. $R = \{(x, y): x = y\} = \{(2, 2), (3, 3)\}$ c. $R = \{(x, y): x = 2y\} = \phi$

d. $R = \{(x, y): x = \frac{y}{2}\} = \{(1, 2), (2, 4)\}$ e. $R = \{(x, y): x = y^2\} = \phi$

4. A × B = {(6, 2), (6, 4), (6, 6), (7, 2), (7, 4), (7, 6), (8, 2), (8, 4), (8, 6), (10, 2), (10, 4), (10, 6)}

i. $R = \{(x, y): x + y < 12\} = \{(6, 2), (6, 4), (7, 2), (7, 4), (8, 2)\}$

ii. $R = \{(x, y): 2x + y > 10\} = \{(6, 2), (6, 4), (6, 6), (7, 2), (7, 4), (7, 6), (8, 2), (8, 4), (8, 6), (10, 2), (10, 4), (10, 6)\}$

Show the representation to the teacher.

6. a. A × A = {(4, 4), (4, 5}, (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)}

b. $R_1 = \{(x, x): x \in A\} = \{(4, 4), (5, 5\}, (6, 6)\}$

c. $R_2 = \{(x, y): (x, y) \in R$ implies $(y, x) \in R\} = \{(4, 5), (5, 4\}, (5, 6), (6, 5)\}$

d. $R_3 = \{(x, y): (x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R\} = \{(4, 5), (5, 6), (4, 6)\}$

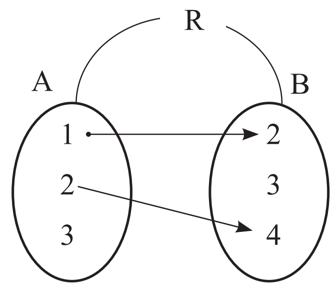
1.1.4 Domain, Range and Co-domain of Relation

Here, R be a relation from set A to B.

The set of the first elements of all ordered pairs in R is called the domain of R.

Thus, domain of R = $\{x \in A: (x, y) \in R\}$

For example: If A = {1, 2, 3}, B = {2, 3, 4} and R=$\{(x, y): y = 2x\}$ ={(1, 2), (2, 4)}. then, domain of R={1, 2}.



Similarly, the set of the second elements of all ordered pairs in R is called the range of R.

Thus, range of R= $\{y \in B: (x, y) \in R\}$

Hence, range of R= {2, 4}.

Co-domain of Relation

Suppose R be a relation from set A to B. The set B is called the co-domain of the relation R. The range of R is always a subset of its co-domain.

For example, if R is a relation from A = {1, 2, 3} to B = {2, 3, 4}, then the co-domain of R = set B ={2, 3, 4}.

Example 1

If R = $\{(x, y): y = 3x; x \in \{1, 2, 3\}\}$ then

a. List all the ordered pairs in the relation R.

b. Represent it in a mapping diagram.

c. Write the domain and range of the relation R.

Solution

a. Given relation: $y = 3x; x \in \{1, 2, 3\}$

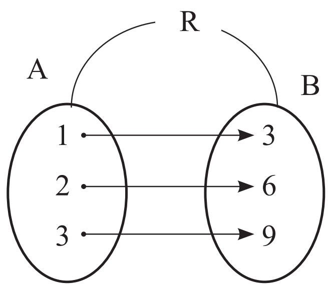
When $x = 1 \rightarrow y = 3 \times 1 = 3$

When $x = 2 \rightarrow y = 3 \times 2 = 6$

When $x = 3 \rightarrow y = 3 \times 3 = 9$

Therefore, R = {(1, 3), (2, 6), (3, 9)}

b. Representing in mapping diagram,



c. Domain of R = {1, 2, 3}

Range of R = {3, 6, 9}

Example 2

Find relation R, in the set of natural numbers N defined by R= $\{(x, y): y = x + 3, x < 4, \text{ and } x, y \in N\}$. Also, find the domain and range of that relation.

Solution

Given R = $\{(x, y): y = x + 3, x < 4, x, y \in N\}$.

The domain of R : $\{x : x < 4, x \in N\} = \{1, 2, 3\}$

Here, $x = 1, 2, 3$ and $y = x + 3$.

when $x = 1 \quad \Rightarrow \quad y = 1 + 3 = 4$,

when $x = 2 \quad \Rightarrow \quad y = 2 + 3 = 5$,

when $x = 3 \quad \Rightarrow \quad y = 3 + 3 = 6$.

Therefore, R= {(1, 4), (2, 5), (3, 6)}.

Hence, domain of R = {1, 2, 3}

Range of R = {4, 5, 6}

Example 1.1 (D)

1. a. Define domain and range of a relation with examples.

   b. The relation R is represented in the table below. Write the domain and range of the relation R

x	1	3	5
y	5	7	9

   c. If R = $\{(a, b), (c, d)\}$, find the domain and range of the relation R.

2. Find the domain and range of the following relation:

   a. R = {(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)}

   b. R = {(2, 4), (2, 6), (3, 6), (3, 9), (4, 8), (4, 12)}

   c. R = {(3, – 1), (4, – 2), (5, – 3), (6, – 4)}

d. R = {(4, – 2), (4, 2), (1, –1), (1, 1)}

e. R = {(5, 8), (6, 9), (7, 10), (8, 11)}

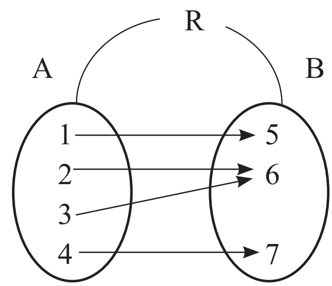
3. Write the range of following relation:

a. $y = 2x, x \in \{0, 1, 5\}$

b. $y = 2x + 3, x \in \{-1, 1, 2\}$

c. $y = 3x^2 - 4, -2 < x \leq 3, x \in W$

4. The relation R from set A to set B is shown in the given figure.



a. Write the relation R in set of ordered pairs.

b. Write the domain and range of the relation R.

5. Find the domain and range of relations given below and present them in set of ordered pairs and mapping diagram.

a. $R = \{(x, y): y = 2x + 1, 0 \leq x \leq 3, x \in W\}$

b. $R = \{(x, y): y = x^2 - 2, 0 \leq x \leq 3, x \in W\}$

c. $R = \{(x, y): y = 3x^2 - 2x - 1, 1 \leq x \leq 4, x \in W\}$

d. $R = \{(x, y): y = x^2 - x, 0 \leq x \leq 3, x \in W\}$

Answer

1. a. Show to the teacher. b. Domain of R = {1, 3, 5}, Range = {3, 5}
   c. Domain of R = {*a*. *c*}, Range = {*b*, *d*}
2. a. Domain of R = {1, 2, 3}, Range= {3, 5}
   b. Domain of R = {2, 3, 4}, Range= {4, 6, 8, 9, 12}
   c. Domain of R = {3, 4, 5, 6}, Range = {–1, –2, –3, –4}
   d. Domain of R = {1, 4}, Range= {1, –1, 2, – 2}
   e. Domain of R = {5, 6, 7, 8}, Range= {8, 9, 10, 11}
3. a. Range = {0, 2, 10} b. Range= {1, 5, 7} c. Range= {–1, –4, –1, 8, 23}
4. a. R = {(1, 5), (2, 6), (3, 6), (4, 7)}, b. Domain of R = {1, 2, 3, 4}, Range= {5, 6, 7}
5. a. Domain of R = {0, 1, 2, 3}, Range= {1, 3, 5, 7}
   b. Domain of R = {0, 1, 2, 3}, Range= {–2, –1, 2, 7}
   c. Domain of R = {1, 2, 3, 4}, Range= {0, 7, 20, 39}
   d. Domain of R = {0, 1, 2, 3}, Range= {0, 2, 6}

1.1.5 Inverse Relation

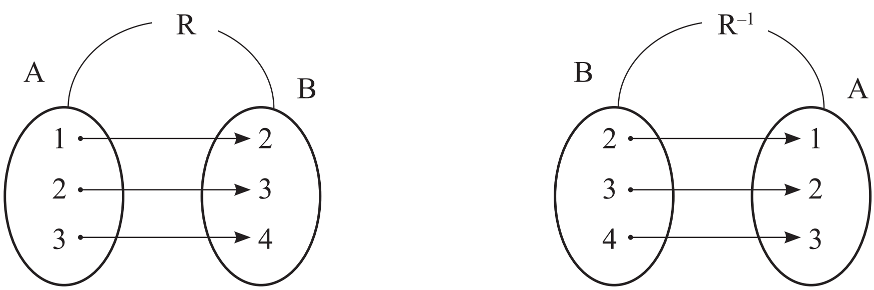
Activity 1

Problem: If A = {1, 2, 3}, B = {2, 3, 4} and R = {(x, y): $y = x + 1$, $x \in$ A}, find the inverse relation of R, and represent it in a mapping diagram.

Procedure: Each student in the class should write R in a group of ordered pairs and present it in a mapping diagram. Make a new set by interchanging the first and the second members of the ordered pairs in the relation R to each other. Also present R^{-1} in a mapping diagram and compare whether the domain and range of R and R^{-1} are the same.

Conclusion: ....

In the mapping diagram



The new relation formed by interchanging the domain and range of a relation R from a non-empty set A to a non-empty set B is called the inverse relation of R, denoted by R^{-1}.

In other words, the set of ordered pairs obtained by interchanging the first and second elements of each ordered pair in R is called the inverse relation of R and is denoted by R^{-1}.

Example 1

If A = {1, 2, 3, 4) and R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 4)}

a. Write the inverse relation R as a set of ordered pairs.

b. Write the domain and range of R^{-1}.

Solution

a. R^{-1} = {(1, 1), (2, 2), (3, 3), (4, 4), (3, 2), (4, 3)}

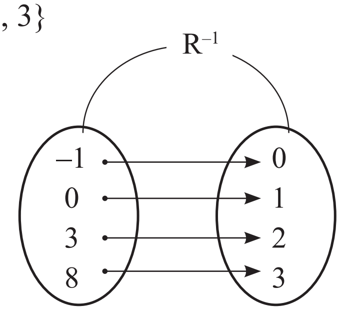
b. Domain of R^{-1} = {1, 2, 3, 4}, and range of R^{-1} = {1, 2, 3, 4}

Example 2

Find the domain and range of the inverse relation of R = $\{(x, y): y = x^2 - 1, 0 \le x \le 3, x \in W\}$.

Solution

Here, domain of R = $\{x: 0 \le x \le 3, x \in W\} = \{0, 1, 2, 3\}$



When, $x = 0$, $y = x^2 - 1 = 0^2 - 1 = -1$

When, $x = 1$, $y = x^2 - 1 = 1^2 - 1 = 0$

When, $x = 2$, $y = x^2 - 1 = 2^2 - 1 = 3$

When, $x = 3$, $y = x^2 - 1 = 3^2 - 1 = 8$

R = {(0, –1), (1, 0), (2, 3), (3, 8)}

R^{-1} = {(–1, 0), (0, 1), (3, 2), (8, 3)}

Domain of R^{-1} = {–1, 0, 3, 8}range of R^{-1} = {0, 1, 2, 3}

Exercise 1.1 (E)

1. a. What do you mean by inverse relation? Write with an example.

   b. If R = $\{(a, b), (c, d)\}$ write R^{-1}.

2. Write the inverse relations of given relations:

   a. R = {(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)}

   b. R = {(1, 3), (1, 6), (2, 3), (2, 6), (3, 3), (3, 6)}

   c. R = {(3, – 1), (4, – 2), (5, – 3), (6, – 4)}

   d. R = {(8, 6), (7, 5), (6, 4), (5, 3), (4, 2), (3, 1)}

3. Write the ordered pairs of relation given below and find their inverse relation.

   a. R = $\{(x, y): y = 3x, x \in \{1, 2, 3\}\}$

   b. R = $\{(x, y): y = 2x + 3, x \in \{1, 2, 3\}\}$

   c. R = $\{(x, y): y = x - 2, x \in \{6, 7, 8\}\}$

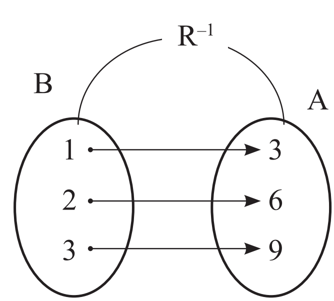
   d. R = $\{(x, y): y = x^2, x \in \{0, 1, -1\}\}$

4. The given figure shows the relation R^{-1} from set B to A.

   a. Write the relation R^{-1} as a set of ordered pairs, also write the domain and range.

   b. Write the domain and range of the relation R.

   c. Write R^{-1} and R according to the set builder method.



5. Suppose set A = {2, 3, 4, 5, 6, 7, 8, 9}, relation R = {(x, y): y = multiple of x}. Write R as a set of ordered pairs and find the followings:
   a. Domain and range of R b. Inverse relation (R^{-1}) of R
   c. Domain and range of R^{-1}
6. If A = {2, 3, 4, 5} and the relation R defined on A = {(x, y): x is a factor of y}, find the relation R expressed as an ordered pairs.
   a. Domain of R
   b. Range of R
   c. Find the inverse relation R^{-1} and present it in a mapping diagram.
7. Find the inverse of the following relations, and write the domain and range. Also, present them in a mapping diagram:
   a. $\{(x, y): y = x - 1, 4 \le x \le 7, x \in N\}$
   b. $\{(x, y): y = x^2 - 1, 0 \le x \le 3, x \in W\}$
   c. $\{(x, y): y = 3x^2 - 2x - 1, 1 \le x \le 4, x \in W\}$

Answer

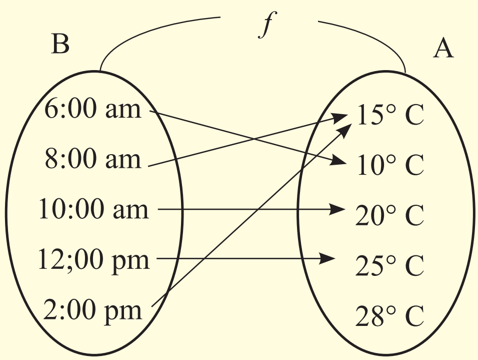
1. Show to the teacher.
2. a. R^{-1} = {(3, 1), (5, 1), (3, 2), (5, 2), (3, 3), (5, 3)} b. R^{-1} = {(3, 1), (6, 1), (3, 2), (6, 2), (3, 3), (6, 3)}
   c. R^{-1} = {(– 1, 3), (– 2, 4), (– 3, 5), (– 4, 6)} d. R^{-1} = {(6, 8), (5, 7), (4, 6), (3, 5), (2, 4), (1, 3)}
3. a. R^{-1} = {(1, 3), (2, 6), (3, 9)} b. R^{-1} = {(1, 5), (2, 7), (3, 9)}
   c. R^{-1} = {(6, 4), (7, 5), (8, 6)} d. R^{-1} = {(0, 0), (1, 1), (–1, 1)}
4. a. R^{-1} = {(1, 3), (2, 6), (3, 9)}, Domain of R^{-1} = {1, 2, 3}, Range of R^{-1} = {3, 6, 9}
   b. Domain of R = {3, 6, 9}, Range of R = {1, 2, 3} c. R^{-1} = {(x, y), y = 3x} R = {(x, y), $y = \frac{x}{3}$ }
5. R = {(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)},
   a. Domain of R = {2, 3, 4, 5, 6, 7, 8, 9}, Range of R = {2, 3, 4, 5, 6, 7, 8, 9}
   b. R^{-1} = {(2, 2), (4, 2), (6, 2), (8, 2), (3, 3), (6, 3), (9, 3), (4, 4), (8, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}
   c. Domain of R^{-1} = {2, 3, 4, 5, 6, 7, 8, 9}, Range of R^{-1} = {2, 3, 4, 5, 6, 7, 8, 9}
6. R = {(2, 2), (2, 4), (3, 3), (4, 4), (5, 5)},
   a. Domain of R = {2, 3, 4, 5}, Range of R = {2, 3, 4, 5}
   b. R^{-1} = {(2, 2), (4, 2), (3, 3), (4, 4), (5, 5)}
   c. Domain of R^{-1} = {2, 3, 4, 5}, Range of R^{-1} = {2, 3, 4, 5}
7 a. R^{-1} = {(3, 4), (4, 5), (5, 6), (6, 7)}, Domain of R^{-1} = {3, 4, 5, 6}, Range of R^{-1} = {4, 5, 6, 7}
   b. R^{-1} = {(–1, 0), (0, 1), (3, 2), (8, 3)}, Domain of R^{-1} = {–1, 0, 3, 8}, Range of R^{-1} = {0, 1, 2, 3}
   c. R^{-1} = {(0, 1), (7, 2), (20, 3), (39, 4)}, Domain of R^{-1} = {0, 7, 20, 39}, Range of R^{-1} = {1, 2, 3, 4}
   Show the mapping diagram to the teacher.

1.1.6 Function

Activity 1

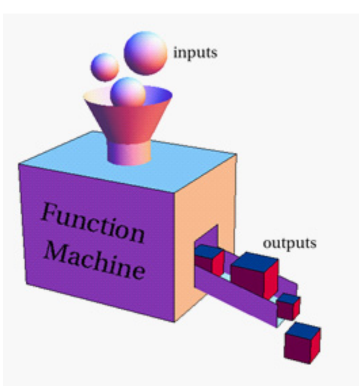
Problem: Study the relation between the elements of the two sets given below. The first set represents time and the second set represents temperature (°C). The temperature of a city at different times of the day is provided. What would you call such a relationship? Discuss and draw conclusions.

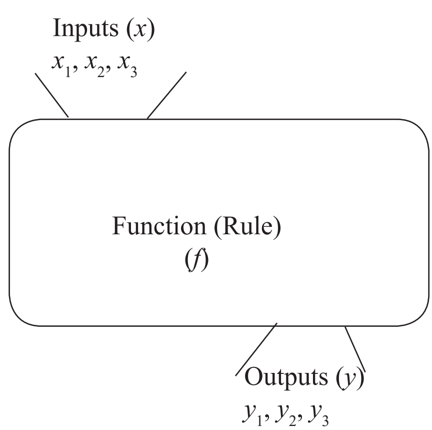
Procedure: Each student should observe whether there is only one temperature corresponding to each time or not, and draw conclusions with the help of the teacher.



By studying the above sets, it is observed that each time in the first set is related to only one temperature in the second set. That is, there is no relation where two different temperatures correspond to the same time. Thus, such a specific relation between two variables of time and temperature is called a function.

Study the figure given below:



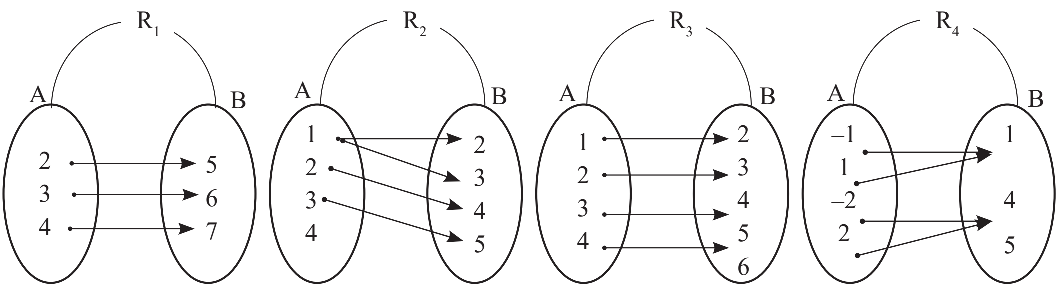


In the figure above, each input has only one output. So, in a function, every input gives exactly one output.

A special relation in which every element of the first set is related to exactly one element of the second set is called a function. If a relation is defined from set A to set B, where A and B are two non-empty sets, and every element of A has a unique (exactly one) image in B, then that relation is called a function. It is denoted by $f: A \to B$ and written as $y = f(x)$, where $x \in A$ and $y \in B$. It is read as "y is a function of x."

Example 1

Study the mapping diagrams below, and determine whether each relation is a function or not.



Among the given relations R_1, R_3, and R_4, each element of set A is associated with only one element of set B. These relations are function. But relation R_2 is not a function because 4 has no association. Also element 1 in R_2 has two images; 2 and 3.

Example 2

Which of the following relations are function? Write with reason.

a. $R_1 = \{(2, 3), (3, 4), (5, 6)\}$

b. $R_2 = \{(2, 3), (2, 5), (5, 6)\}$

c.

Input (x)	0	1	2	0
Output (y)	–4	–2	0	4

Solution

a. R_1 is a function because each element in the first set is associated with exactly one element in the second set.

b. R_2 is not a function because element 2 of first set is associated with two elements 3 and 5 of second set, so it is not a function.

c. The relation is not a function because 0 is associated with –4 and 4.

Example 3

Write the meaning of $y = f(t)$.

Solution

$y = f(t)$ means that y is a function of t. Here, by operating on t, the value of y is obtained. Here, y is the dependent variable, and t is the independent variable.

Example 4

Express the area of a square a. as a function of its perimeter (P). Also, find the area of a square whose perimeter is 8 m.

Solution

The perimeter of a square is given by $P = 4l$ where l is the length of sides of the square. So, $l = \frac{P}{4}$

Now, the area of the square, $A = l^2 = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$

Thus, $A = f(P)$, where $f(P) = \frac{P^2}{16}$ [$\because$ the area is dependent on the perimeter.]

Again, the perimeter (P) = 8m, then the area, $A = f(P) = \frac{P^2}{16} = \frac{8^2}{16} = 4\ m^2$.

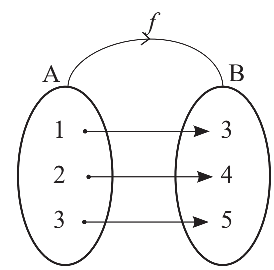
Representation of Function

Functions can be represented by the different ways which are given below.

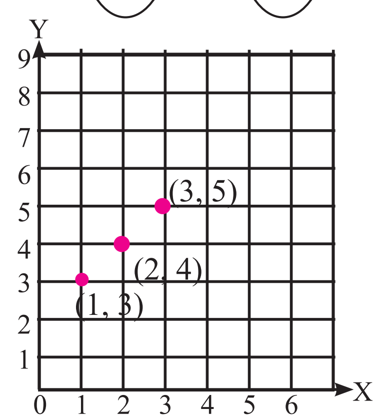
a. Visual Representation

i. Arrow or Mapping Diagram

Set A = {1, 2, 3} and Set B= {3, 4, 5} and if f(1) = 3, $f(2) = 4$, $f(3) = 5$, then the function f can be represented in arrow diagram as follows:



ii. Graph: According to this method, the relation between the first set and the second set of the function is represented by forming ordered pairs from the members of these sets. These ordered pairs are then plotted on a graph. From the above example, the ordered pairs are (1, 3), (2, 4), and (3, 5) respectively, and these are represented in the graph.



b. Numerical Representation

i. **Set of Ordered Pairs:** From the above example, the function 'f' can be represented as a set of ordered pairs as follows:

$f = \{(1, 3), (2, 4), (3,5)\}$

ii. **Table:** In this method, the members of the first set and the members of the second set associated with them are presented in a table. The above function f is in the presented table.

x	1	2	3
$y = f(x)$	3	4	5

c. Verbal Representation

A function can be represented verbally, which describes the relation between the input variable and the output variable. For this, the relation between the members of the two sets needs to be known. For example, describing the function $f = \{(1, 3), (2, 4), (3, 5)\}$ verbally: for every value of x, the value of $f(x)$ is 2 more than x.

d. Algebraic Representation

This is the most commonly used method to define a function. In this method, the relation between x and $f(x)$ of a function $y = f(x)$ is represented by an algebraic expression. In the above example, the value of $f(x)$ is 2 more than the value of x, so $f(x) = x + 2$.

Example 5

Represent the function $f = \{(2, 1), (4, 2), (6, 3)\}$ in a table, using an algebraic expression, and in a graph.

Solution

Given function, $f = \{(2, 1), (4, 2), (6, 3)\}$

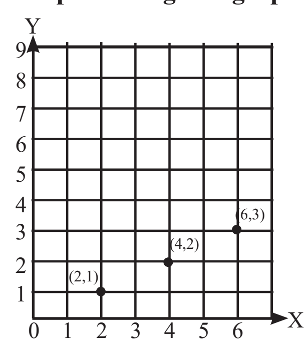
Representing in a table:

x	2	4	6
y	1	2	3

Representing algebraically:

Here, expressing y as a function of x, the value of y is half the value of x, therefore: $y = f(x) = \frac{x}{2}$

Representing in a graph



Vertical Line Test

Relations can also be represented in graphs. But the graph of all relations are not functions. If each element of the domain is related to only one element of the co-domain, then such a relation is called a function. That is, in a function, no single value of x will be associated with two or more values of y. Using this concept, we can use the vertical line test to determine whether a given graph is the graph of a function or not. The vertical line test is applied as follows:

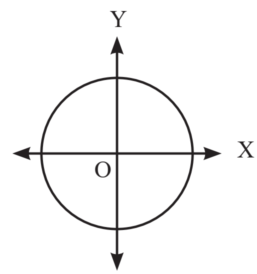
a. Draw several vertical lines perpendicular to the X- axis across the region of the given graph. If each of these vertical lines intersects the given graph at only one point, then that graph represents of a function.

b. Draw several vertical lines perpendicular to the X- axis across the region of the given graph. If any one of these vertical lines intersects the given graph at two or more points, then that graph does not represent a function.

In this section, the domain of all graphs is assumed to the intervals of real number. The study of interval will be studied in the next chapter.

Example 5

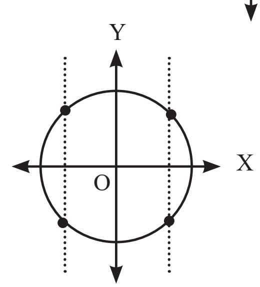
State with reason whether the given graph alongside is graph of a function or not.



Solution

Applying the vertical line test:

In the above text, the vertical lines drawn perpendicular to X- axis intersect the given graph at two points. Therefore, the given graph is not a graph of a function.



Exercise 1.1 (F)

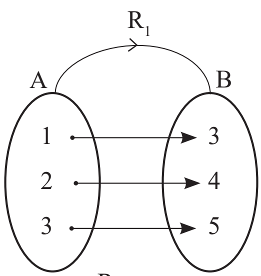
1. Define a function with an example used in daily life.
2. a. Write the meaning of the functional notation $y = f(t)$.
   b. Write the meaning of the functional notation $P = f(D)$, where D represents the density of water and P represents pressure.
   c. Write the meaning of the functional notation $y = f(x)$.
3. a. Express the perimeter of a square (P) as a function of its area (A).

b. Express the area of a circle (A) as a function of its circumference (C).

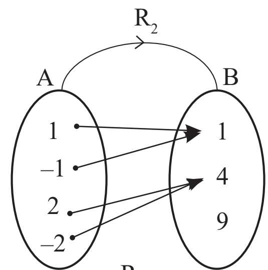
c. Express the volume of a cube (V) as a function of its surface area (A).

4. State whether the following relations are functions or not with reason.

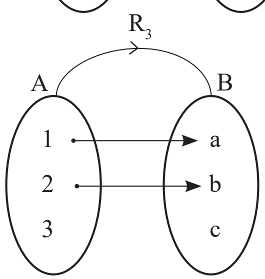
a.



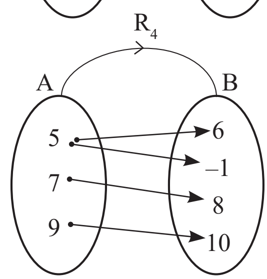
b.



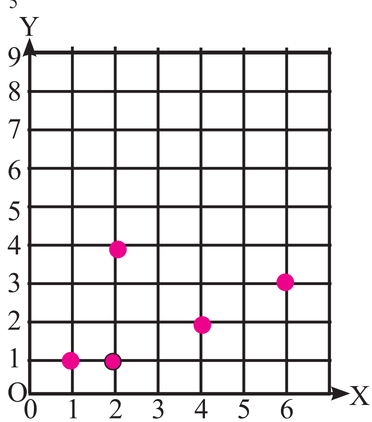
c.



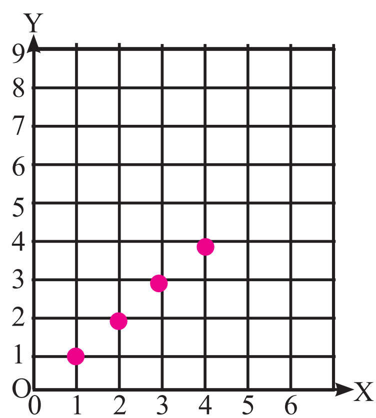
d.



e. R_5



f. R_6



g. R_7 = {(2, 4)}, (3, 6), (4, 8)}

h. R_8 = {(3, 6), (2, 4)}, (3, 9), (4, 16)}

i. R_9

x	3	4	5
y	2	3	4

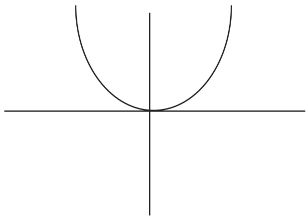
j. R_{10}

x	2	1	2
y	7	6	3

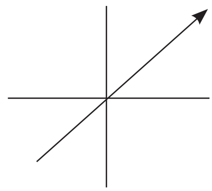
5. Write all the functions from question no. 4 in algebraic form.

6. State whether the following graphs are graph of functions or not with reason.

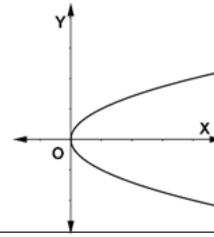
a.



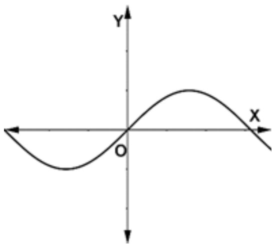
b.



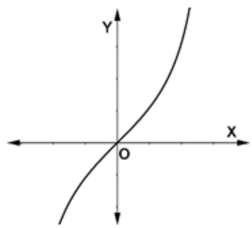
c.



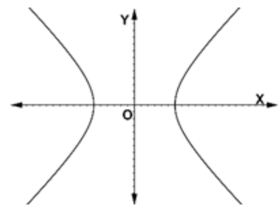
d.



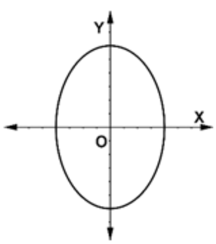
e.



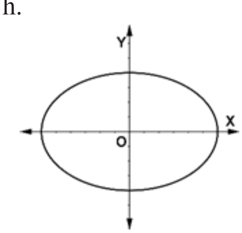
f.



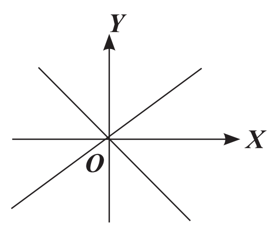
g.



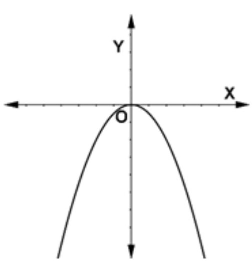
h.



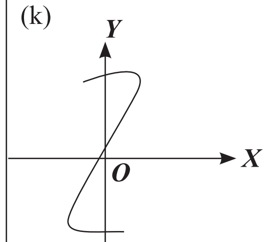
i.



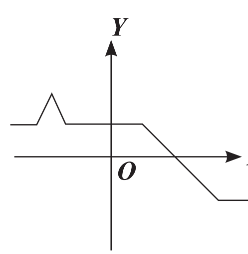
(j)



(k)



(*l*)



7. Express the given functions in the table, algebraic representation and graph.

a. $f_1 = \{(1, 1), (2, 4), (3, 9)\}$

b. $f_2 = \{(1, 5), (2, 7), (3, 9)\}$

c. $f_3 = \{(3, 1), (5, 3), (7, 5)\}$

d. $f_4 = \{(0, 3), (1, 4), (2, 5)\}$

Creative Activity

The price of an apple is Rs. 10. If a shopkeeper sells x apples, how much money will he/she earn?

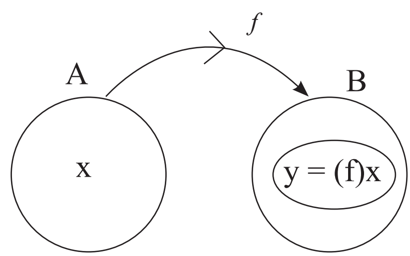
1. Represent this problem by taking the earning as I.
2. Draw a graph by putting the values of x.
3. If the shopkeeper wants to earn more than Rs. 8000, what should he/she do? Give an answer covering as many possibilities as possible.

Answer

1. Show to the teacher.
2. a. $y = f(t)$ means y is a function of t. b. $P = f(D)$ means P is a function of D.
   c. $y = f(x)$ means y is a function of x.
3. a. $P = 4\sqrt{A}$ b. $A = \frac{C^2}{4\pi}$ c. $V = \sqrt{\frac{A^3}{36\pi}}$
4. a. R_1 is a function because all members of the first set are associated with only one member of the second set.
   b. R_2 is a function because all members of the first set are associated with only one member of the second set.
   c. R_3 is not a function because the member 3 of the first set is not associated with any member of the second set.
   d. R_4 is not a function because the member 5 of the first set is associated with two members (7, 9) of the second set.
   e. R_5 is not a function because $x = 2$ is associated with two members (1 and 4) of the second set.
   f, R_6 is a function because all members of the first set are associated with only one member of the second set.
   g, R_7 is a function because all members of the first set are associated with only one member of the second set.
   h. R_8 is not a function because not all members of the first set are associated with only one member of the second set (e.g., 3 is associated with 6 and 9).
   i. R_9 is a function because all members of the first set are associated with only one member of the second set.
   j. R_{10} is not a function because not all members of the first set are associated with only one member of the second set (e.g., 2 is associated with 7 and 3).
5. $y = f_1(x) = x + 2$, $y = f_2(x) = x^2$, $y = f_6(x) = x$, $y = f_7(x) = 2x$, $y = f_9(x) = x - 1$
6. a. a function
   b. a function c. not a function d. a function e. a function
   f. not a function g. not a function h. not a function i. not a function
   j. a function k. not a function l. a function
7. Show to the teacher.

1.1.7 Domain, Co-domain and Range, Image and Pre-image of a Function

In the function f: A → B, set A is called the Domain, and set B is called the Co-domain. Similarly, the set of members of set B that are related to the members of set A is called the range. The members of set B are associated with set A. They are called the Image of f, and the corresponding member of A is called the pre-image.

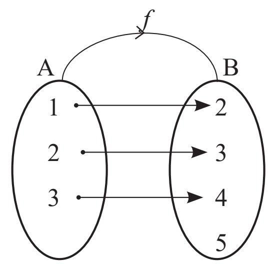


For example: If set A = {1, 2, 3}, set B = {2, 3, 4, 5} and f = {(1, 2), (2, 3), (3, 4)} then:

Domain of function f = {1, 2, 3}

Co-domain of function f = {2, 3, 4, 5}

Range of function f = {2, 3, 4}



Hence, range of a function is always a subset of the co-domain.

The image of 1 is 2, or the pre-image of 2 is 1. This is written as $f(1) = 2$. Similarly, $f(2) = 3$, $f(3) = 4$.

Example 1

If the function f = {(1, 2), (–2, –4), (3, 6), (–4, –8)}, find the domain and range of f.

Solution

Given function, f = {(1, 2), (–2, –4), (3, 6), (–4, –8)}

Domain of f = Set of the first elements of the ordered pairs = {1, –2, 3, –4}

Range of f = Set of the second elements of the ordered pairs = {2, –4, 6, –8}

Example 2

If the range of the function $f(x) = 7x - 8$ is 13, what is its domain? Find it.

Solution

Given function, $f(x) = 7x - 8$, Range = 13, therefore $f(x) = 13$

Here, $f(x) = 13$

Or, $7x - 8 = 13$

Or, $7x = 21$, so $x = 3$

Therefore, domain = {3}

Example 3

If f: A → B where A = {–1, 0, 1, 2} and $f(x) = 2x + 1$, find the range of f and represent it in arrow diagram.

Solution

Here, $f(x) = 2x + 1$

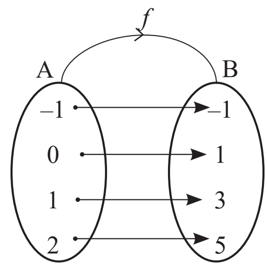
$f(-1) = 2 \times (-1) + 1 = -2 + 1 = -1$

$f(0) = 2 \times 0 + 1 = 0 + 1 = 1$

$f(1) = 2 \times 1 + 1 = 2 + 1 = 3$

$f(2) = 2 \times 2 + 1 = 4 + 1 = 5$

Therefore, the range of f is {–1, 1, 3, 5}.



Example 4

If the function $f(x) = 2x + 3$, then which element has the image 7? Find it.

Solution

Here, let, 7 is image of x. So, $f(x) = 7$, x = ?

$f(x) = 7$

or, $2x + 3 = 7$ or, $2x = 4$, $x = 2$

Therefore, the image of 2 is 7.

Example 5

The table alongside shows the number of pens and its price. Answer the following questions.

a. If the number of pens is denoted by x and the total price by y, express the relation between these two variables in the form of a function as a formula.

b. Write the domain of the function.

c. Write the range of the function.

Number of pens (x)	Price (y)
1	10
2	20
3	30
4	40

Solution

a. Since the number of pens be x and the price be y:

When, $x = 1, y = 10 = 10 \times 1$

When, $x = 2, y = 20 = 10 \times 2$

When, $x = 3, y = 30 = 10 \times 3$

When, $x = 4, y = 40 = 10 \times 4$

Therefore, $y = 10 \times x$

So, the function 'f' representing the price in terms of the number of pens is $f(x) = 10x$.

b. Domain of function f = {1, 2, 3, 4} c. Range of function f = {10, 20, 30, 40}

Exercise 1.1 (G)

1. a. Explain the domain, co-domain and range of a function with an example.

   b. Explain the image and pre-image of a function with an example.

2. The following table represents the function f.

x	1	2	3	4
y	1	4	9	16

   a. Write the function f as a set of ordered pairs.

   b. Write the domain and range of f.

   c. What is the pre-image of 9? Write it.

   d. Write the function f in algebraic form.

3. Answer the quesstions based on the given mapping diagram:

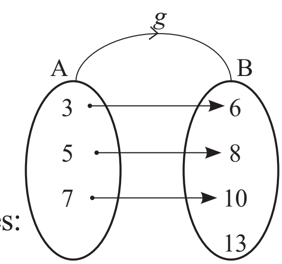
   a. Does g represent a function? Write with reason.

   b. What is the pre-image of 8?

   c. Write the domain, co-domain and range of g.

   d. What is the value of $g(5) + g(7)$?

   e. Write the function g in algebraic form.



4. Find the range of the function in each of the following cases:

   a. $f = \{(2, 4), (3, 6), (4, 8)\}$

   b. $g = (x, y): y = x - 3, 2 < x < 5, x \in N \}$

   c. $h(x) = x^2 - 2x$, Domain = $\{0, 1, 2\}$

5. Find the domain of the function in each of the following cases:

   a. $f(x) = 3x + 5$, Range = $\{2, 8\}$

   b. $g(x) = 4x - 5$, Range = $\{-1, 7\}$

   c. $h(x) = x^2$, Range = $\{9, 16\}$

   d. If the set of images of the function $g(x) = 2x + 5$ is $\{7, 11, 15\}$, find the set of pre-images.

   e. If the set of images of the function $f(x) = 4x - 3$ is $\{1, 5, 9\}$, find the set of pre-images.

6. Answer the following questions:

   a. For which element in the domain of the function $f(x) = \frac{(2x-3)}{5}$ is the image 7? Find.

   b. For which element in the domain of the function $f(x) = 3x + 5$ is the image 8? Find.

c. Which element of the domain has image $\frac{1}{2}$ in the function $f(x) = \frac{2}{x-4}$?

d. Which element of the domain has image 9 in the function $f(x) = 2x - 5$?

7. If a function f: A → W, where A = {$0 \le x \le 4$, $x \in$ W}, is defined by $f(x) = 2x - 3$.

a. Write the domain of the function.

b. Find the range of the function.

c. Represent the function in a arrow diagram and state the type of function.

Answer

1. Show to the teacher.
2. a. f = {(1, 1), (2, 4), (3, 9), (4, 16)}

   b. D = {1, 2, 3, 4}, R = {1, 4, 9, 16} c. 3 d. $f(x) = x^2$
3. a. g represents a function because every element of the domain has a unique image.

   b. 5 c. Domain = {3, 5, 7}, Co-domain = {6, 8, 10, 13}, Range = {6, 8, 10}

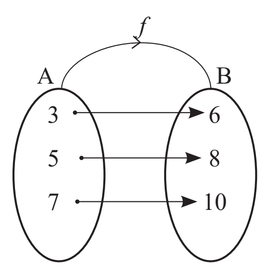
   d. 18 e. $g(x) = x + 3$
4. a. {4, 6, 8} b. {0, 1} c. {0, –1}
5. a. {–1, 1} b. {1, 3} c. {±3, ±4} d. { 1, 3, 5} e. {1, 2, 3}
6. a. 19 b. 1 c. 8 d. 7
7. a. {0, 1, 2, 3, 4}, b.{ –3, –1, 1, 3, 5}

Types of Function

Study the given examples of functions, and discuss them in class.

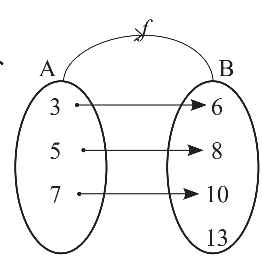
Onto Function

A function f : A → B is said to be an onto function of the range and co-domain are identical or equal. In other words, a function where every element of the co-domain has a pre-image is called an onto function.



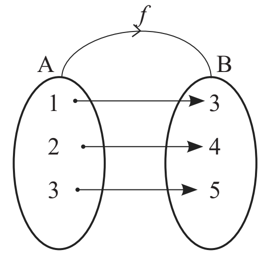
Into Function

A function f : A → B is said to be an into function, if some of the elements of the co-domain don't have a pre-image. In this type of function, the range is a proper subset of the co-domain.



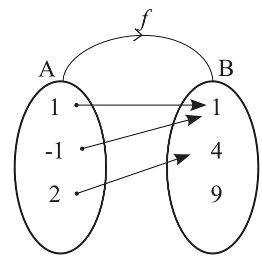
One to One Function

A function 'f' from set A to set B is said to be one to one function, if for every elements in Rance set there is exactly one pre-image in domain set. We use horizontal line test to determine whether a function is one to one.



Many to One Function

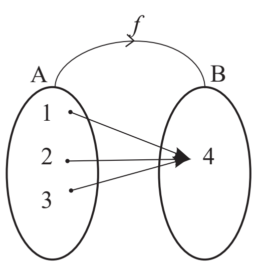
A function $f : A \rightarrow B$ is said to be many to one fuction, if more than one element of the domain is related to a single element in the co-domain. In other words, a function where at least one element of the co-domain has more than one pre-image is called a many-to-one function.

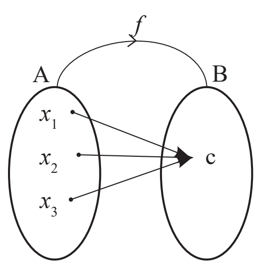


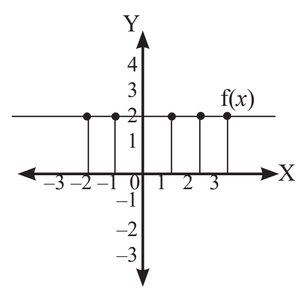
Some Special Types of Functions

Constant Function

A function $f : A \rightarrow B$ where every element of the domain has the same image is called a constant function. A function of the form $f(x) = c$ is called a constant function, where c is a constant.

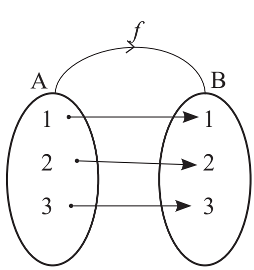


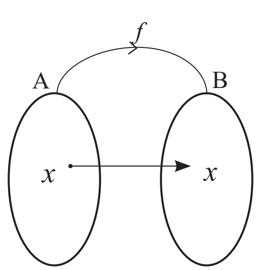


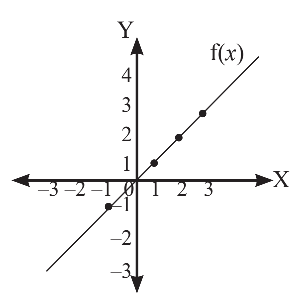


Identity Function

A function $f : A \rightarrow A$ where the image of every element in the domain is itself i.e., the pre-image and image are the same, is called the identity function. Its form is $f(x) = x$.

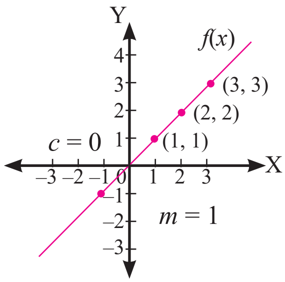


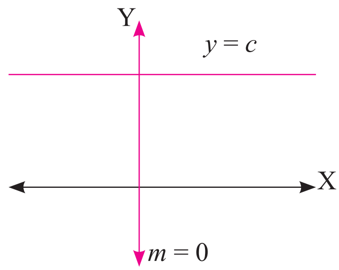


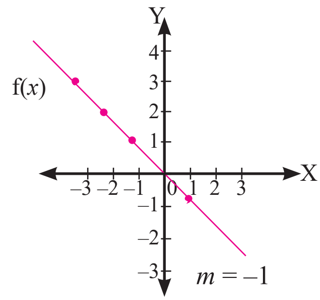


Linear Function

A function of the form $f(x) = mx + c$ is called a linear function. The graph of a linear function is always a straight line. 'm' represents the slope of the line and 'c' represents the y-intercept. The graphs of linear functions for different values of 'm' are shown below.

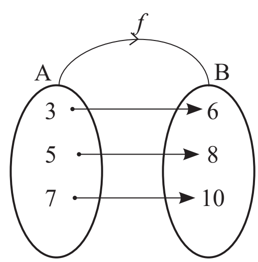






Example 1

The function f is represented by the mapping diagram alongside.
What type of function is this?
Write with reasons.



Solution

The given function f is a one-to-one onto function because each element of the domain has a different image and the range and co-domain are equal.

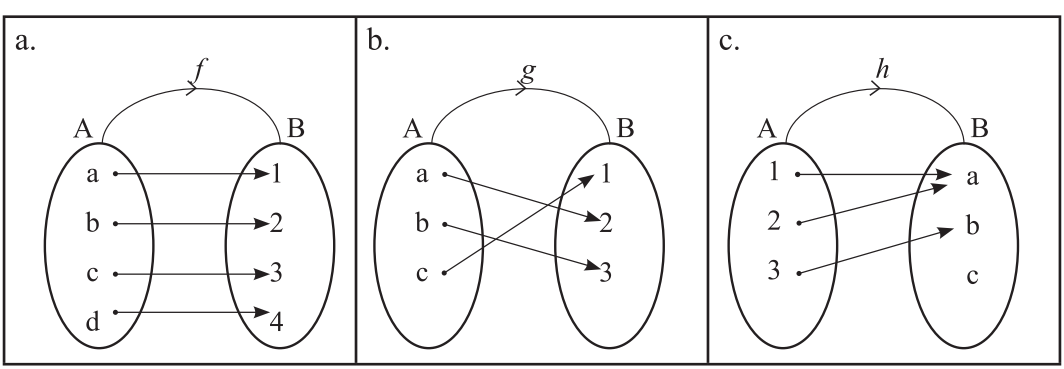
Exercise 1.1 (H)

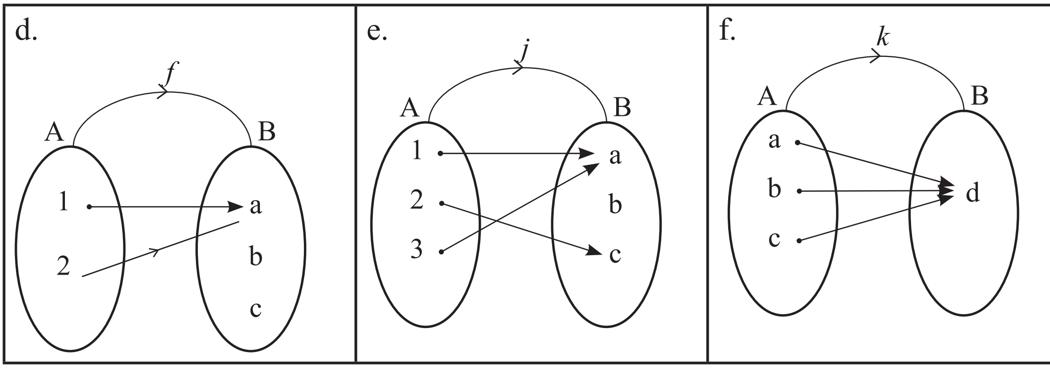
1. Define the following types of functions with examples:

   a. Onto function b. Into function c. Linear function

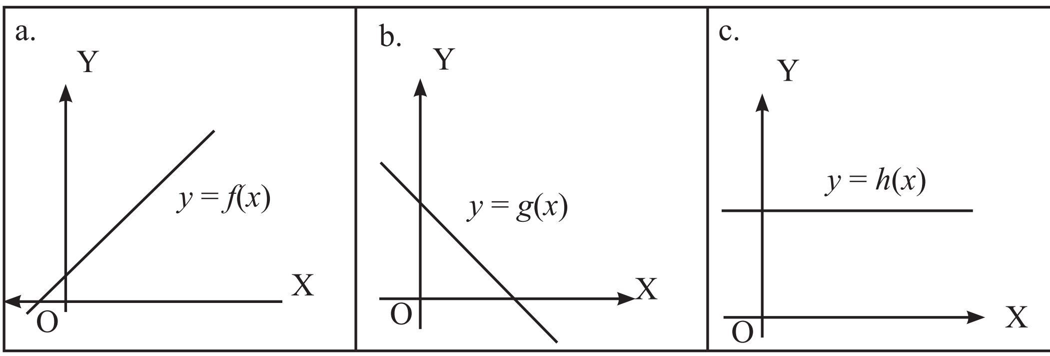
   d. Constant function

2. State the type of each of the given functions with reasons:





3. State the type of function represented by the following graphs:



4 If A = {1, 3}, B = {4, 6, 8}, construct the following functions from A to B:
   a. One-to-one function b. One-to-one into function
   c. Many-to-one into function
5. If P = {3, 4, 5}, Q = {4, 5, 6}, construct the following functions from P to Q:
   a. One-to-one onto function b. Constant function
   c. Many-to-one into function d. Identity function
6. Represent the following functions in mapping diagrams, and state their types:
   a. $f = \{(1, 2), (2, 4), (3, 6)\}$ b. $g = \{(1, 1), (-1, 1), (2, 4), (3, 9)\}$
   c. $h = \{(1, 0), (2, 0), (3, 0)\}$

Answer

1. Show to the teacher.
2. a. Function f is one-to-one onto because different elements of the domain are related to different elements of the co-domain and the range and co-domain are equal.
   b. - f. Show to the teacher.
3. a. Linear, b. Linear, c. Constant

4 - 5. Show to your teacher.

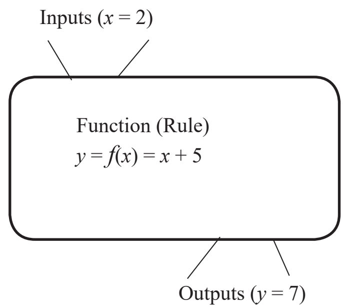
6. a. One-to-one onto, b. Many-to-one onto, c. Constant function

1.1.8 Problems Related to Function

Functional value

If $f(x)$ is a function and for a given value of x, the corresponding value y is obtained, then the value y thus obtained is called the functional value. For example,

if $f(x) = x + 5$ and $x = 2$, then the value of the function is $f(2) = 2 + 5 = 7$. $f(x)$ is called the image of x, and x is called the pre-image of y or $f(x)$.



Example 1

If the function is $f(x) = 2x - 4$, find the functional values $f(1)$, $f(-1)$, $f(2)$, $f(-2)$.

Solution

Given, $f(x) = 2x - 4$

$f(1) = 2 \times 1 - 4 = -2$

$f(-1) = 2 \times (-1) - 4 = -6$

$f(2) = 2 \times (2) - 4 = 0$

$f(-2) = 2 \times (-2) - 4 = -8$

$\therefore f(1) = -2,\ f(-1) = -6,\ f(2) = 0,\ f(-2) = -8$

Example 2

If $f(x + 3) = 2x + 1$, find $f(x)$ and $f(-3)$.

Solution

Given, $f(x + 3) = 2x + 1, f(x + 3) = 2(x + 3) - 5$

So, replacing $x + 3$ by x,

Therefore, $f(x) = 2x - 5$

And, $f(-3) = 2 \times (-3) - 5 = -6 - 5 = -11$

Alternative Method

Let, $x + 3 = a$,

Then, $x = a - 3$

$f(x + 3) = 2x + 1$

So, $f(a) = 2(a - 3) + 1$

Therefore, $f(a) = 2a - 5$

Therefore, $f(x) = 2x - 5$ (replacing a by x)

Example 3

If $f(x) = 2x$, show that $f(a + b) = f(a) + f(b)$.

Solution

Given, $f(x) = 2x,\ f(a) = 2a,\ f(b) = 2b, f(a + b) = 2(a + b)$

Now, L.H.S. = $f(a + b) = 2(a + b) = 2a + 2b = f(a) + f(b)$ = R. H. S.

Example 4

If a function f:R $\to$ R is defined by $f(3x) = f(3) + f(x)$, find the values of $f(1)$, $f(3)$ and $f(9)$.

Solution

Here, $f(3x) = f(3) + f(x)$

Putting $x = 1$, $f(3 \times 1) = f(3) + f(1)$

Or, $f(3) = f(3) + f(1)$

Or, $f(1) = f(3) - f(3) = 0$

$\therefore$ $f(1) = 0$

Putting, $x = 0$, $f(3 \times 0) = f(3) + f(0)$

Or, $f(0) = f(3) + f(0)$

Or, $f(3) = f(0) - f(0) = 0$

$\therefore$ $f(3) = 0$

Putting $x = 3$, $f(3 \times 3) = f(3) + f(3)$

Or, $f(9) = 0 + 0 = 0$

$\therefore$ $f(9) = 0$

Example 5

If the function $f(x) = px + q$, $f(1) = 4$ and $f(3) = 6$, then

a. Find the values of p and q. b. Find the function f (x).

c. Find the value of $f(-3) + f(0)$.

Solution

a. Here, $f(\mathrm{x}) = px + q$, $f(1) = 4$ and $f(3) = 6$

Now, $f(1) = p + q$ and $f(3) = 3p + q$

From the question, $p + q = 4$ … (i)

$3p + q = 6$ … (ii)

Solving equations (i) and (ii), we get $p = 1$ and $q = 3$.

b. Function $f(x) = px + q = x + 3$

c. Here, $f(x) = x + 3$, $f(0) = 0 + 3 = 3$, $f(-3) = -3 + 3 = 0$

Therefore, $f(-3) + f(0) = 0 + 3 = 3$.

Exercise 1.1 (H)

1. a. If $f(x) = 4x + 5$, find the values of $f(2)$, $f(3)$, $f(5)$.

   b. If $f(x) = 2x^2 - 1$, find the values of $f(-1)$, $f(0)$, $f(2)$.

   c. If $h(x) = 3x^2 + 2\mathrm{x} - 1$, find the values of $h(0)$, $h(2)$, $h(4)$.

   d. If $g(x) = x^3 - 2$, find the values of $g(1)$, $g(-1)$, $g(2)$, $g(-2)$.

e. If $f(x) = \frac{(x^2 + 3x - 2)}{(x - 2)}$ $x - 2 \neq 0$ find the value of $f(4) - f(1) + f(0)$.

2. a. If $f(2x + 3) = 2x - 1$, find $f(x)$ and f (-2).
   b. If $f(3x - 4) = 6x - 5$, find $f(x)$ and f (2).
   c. If f $(3x - 5) = 6x - 13$, find $f(5x)$ and $f(-3)$.
   d. If $f(3x + 2) = 12x - 5$, find f (x) and $f(6)$.
3. a. If $f(y) = a^y$, prove that $f(a - b) = f(a) \div f(b)$.
   b. If $f(x) = 5^x$, prove that f $(a + b) = f(a) \times f(b)$.
   c. If $f(m) = 7^m$, prove that $f(xy) = (f(y))^x$.
4. a. If a function f : R → R is defined by $f(5x) = f(5) + f(x)$, find the values of $f(1)$, $f(5)$ and $f(25)$.
   b. If a function $f(x + 3) = f(3) + f(x)$, prove that $f(0) = 0$, $f(-3) = -f(3)$.
   c. If a function $f(x + 7) = f(10) + f(x)$, prove that $f(3) = 0$, $f(-4) = -f(10)$.
   d. If a function $f(x + a) = f(x) + f(a)$, $a \in$ R, prove that $f(0) = 0$, $f(-a) = -f(a)$.
5. a. Find the domain for which the functions $f(x) = 2x^2 - 1$ and g $(x) = 1 - 3x$ are equal?
   b. If functions $f(x) = 3x^2 + 4x - 5$ and $g(x) = 2x^2 - 3x - 17$ are given, find the values of x for which $f(x) = g(x)$.
   c. If functions $h(t) = 5t^2 - 3t + 4$ and $g(t) = t^2 + 4t + 1$ are given and $h(t) = g(t)$, then find the values of t.
6. a. If the function $f(x) = ax + b$ and $f(2) = 2$, $f(-3) = -13$ then
      i. Find the values of a and b.
      ii. Find the function $f(x)$.
      iii. Find the value of $f(\frac{1}{3})$.
   b. If the function $f(x) = mx + c$ and $f(3) = 9$, $f(5) = 13$, find the values of m and c.

Answer

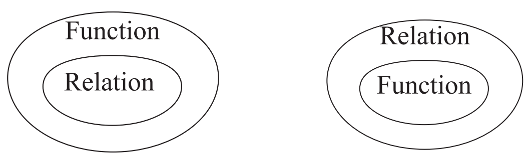
1. a. 13, 17, 25 b. 1, –1, 7 c. –1, 15, 55 d. –1, –3, 6, –10 e. 16
2. a. $x - 4$, –6 b. $2x + 3$, 7 c. $10x - 3$, –9 d. $4x + 3$, 27
4. a. 0, 0, 0 5. a. $\{-2, \frac{1}{2}\}$ b. –3, –4 c. 1, $\frac{3}{4}$
6. a. i. 3, –4 ii. $3x - 4$, iii. –3 b. 2, 3

1.1.9 Difference Between Relation and Function

Let's study the following table.

Relation	Function
A relation is a subset of the Cartesian product of two sets.	A function is a special type of relation in which every element of the domain has a unique association with exactly one element of the co-domain.
A relation between two sets A and B is denoted by R : A $\rightarrow$ B. Here, the domain of the relation can also be a subset of A.	If A and B are considered as domain and co-domain respectively, then a function from A and B is denoted by f : A $\rightarrow$ B. Here, the domain of the function is the set A itself.
Example: R = {(1, 1). (1, –1), (4, 2), (4, –2)}	Example: f = {(1, 1), (2, 4), (3, 9)}

Thought Provoking Question: Which of the following diagrams is correct to indicate the relation between relation and function? Also write the conclusion.



(a) (b)

Example 1

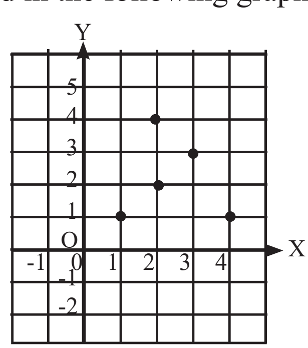
State whether the following relations are functions or not with reason.

a. R = {(1, 2), (2, 3), (2, 4), (3, 5)}

b. R

x	1	2	3	4
y	1	4	9	16

c. Relation R is represented in the following graph:



Solution

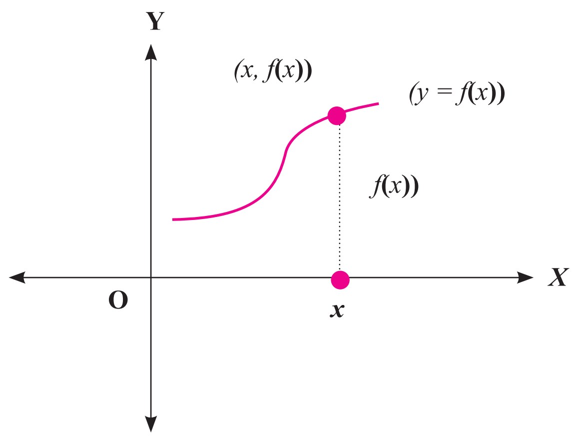
a. Since one element 2 of the domain is associated with 3 and 4. So, the given relation is not a function.

b. Since every element of the domain has a unique association with only one element of the co-domain. So, the given relation is a function.

c. From the graph, the element 2 is associated with both 2 and 4. So, the given relation is not a function.

1.1.10 Graph of the Function $y = x^n$, $n = 1, 2, 3$

The graph of a function $y = f(x)$ is the pictorial representation of the function, which represents the characteristics of the function in a diagramatic form. Many aspects of the function can be analysed, interpreted and predicted using graph of a function. The graph of the function $y = f(x)$ means the set of all ordered pairs $(x, f(x))$ or (x, y) that satisfy the function.



In this section, we will learn to draw the graphs of the functions $y = x$, $y = x^2$ and $y = x^3$.

In this section, the domain and range of all the functions we have discussed are assumed to be the set of real numbers.

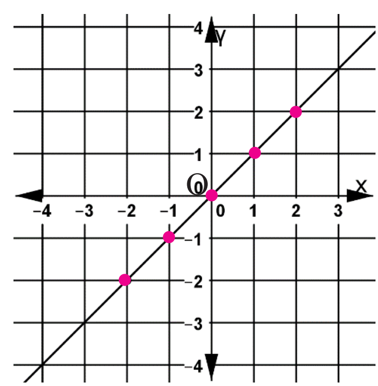
Activity 1

a. Graph of $y = f(x) = x$

Solution

x	–2	–1	0	1	2
y	–2	–1	0	1	2

Plotting the points (–2, –2), (–1, –1), (0, 0), (1, 1), (2, 2) on the graph, we obtain the graph shown on the right.



b. Graph of $y = f(x) = x^2$

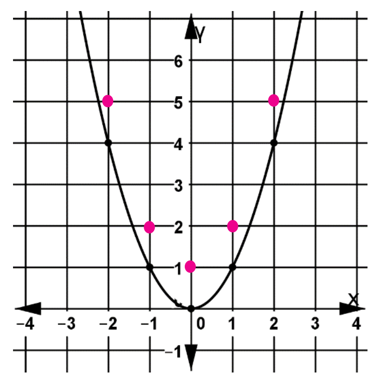
Construct the ordered pairs (x, y) from the given table of x and y values, and plot them on the graph.

x	0	1	–1	2	–2	3	–3
y	...	...	...	...	...	...	...

Let's find some points to plot the graph of the function $f(x) = y = x^2$.

Given function $y = x^2$

x is the independent variable and y is the dependent variable. So, taking some values of x and finding the corresponding values of y, we get the ordered pairs (x, y):



x	–2	–1	0	1	2
y	4	1	0	1	4

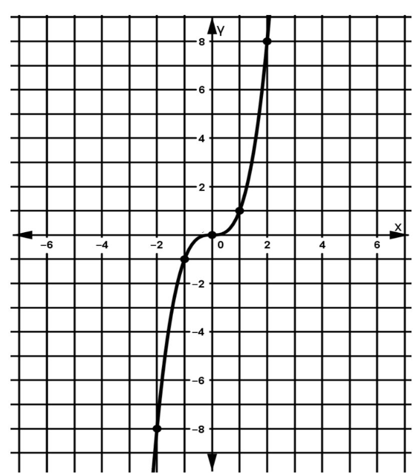
Plotting the points (–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4) on the graph and joining them with a smooth curve we can obtain graph alongside.

Note: The graph of $y = x^2$ is called a parabola. This graph, with turning point or vertex at (0, 0), is called the graph of a quadratic function.

c. Graph of the function $y = f(x) = x^3$

Given function $y = x^3$

x is the independent variable, and y is the dependent variable. So, taking some values of x and finding the corresponding values of y, we get the ordered pairs (x, y):



x	–2	–1	0	1	2
y	–8	–1	0	1	8

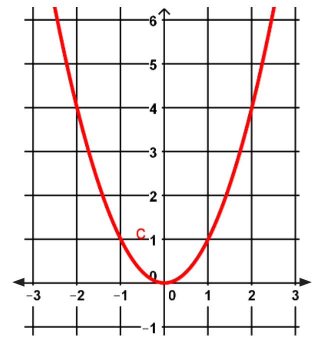
Plotting the points (–2, –8), (–1, –1), (0, 0), (1, 1), (2, 8) on the graph and joining them:

With a smooth curve, we can obtain the graph alongside.

Example 1

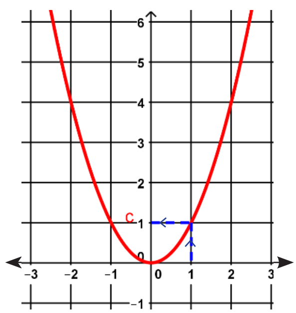
Find the value of y or f (x) based on the specified values of x from the given graph.

x	0	1	2	-1
$y = f(x)$	…	…	…	…

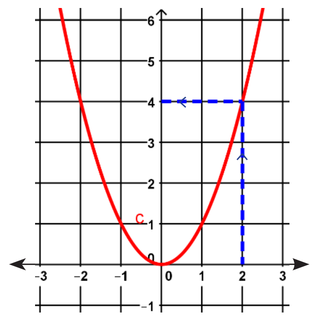


Solution

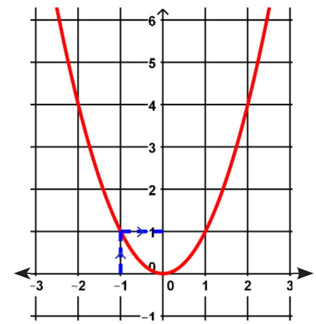
From the graph, when $x = 0$, $f(x) = 0$.



When $x = 1$, $f(x) = 1$



When $x = 2$, $f(x) = 4$



When $x = -1$, $f(x) = 1$

Exercise 1.1 (I)

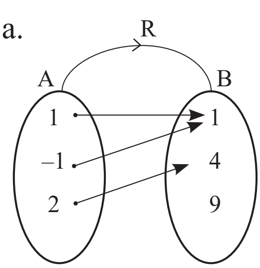
1. State which of the following relations are functions. Write with reason.

   a. R = {(1, 2), (3, 5), (5, 7), (7, 9)}   b. R = {(1, 2), (2, 3), (3, 5), (3, 6)}

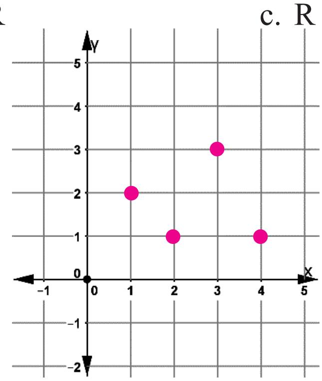
   c. R = {(1, 3), (1, 5), (4, 3)}

2. State which of the following relations are functions. Write with reason.

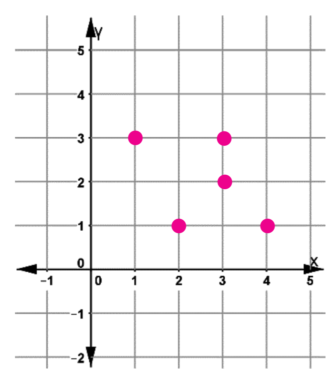
   a.



   b. R



   c. R



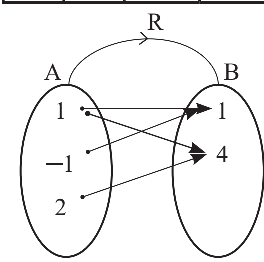
d. R

x	1	2	3
y	1	4	9

e. R

x	1	1	2
y	1	–1	4

f.

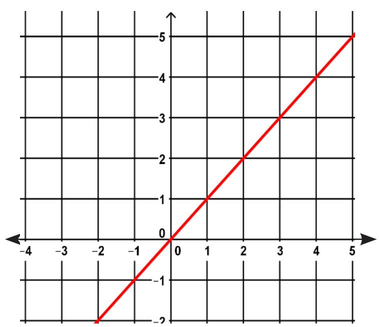


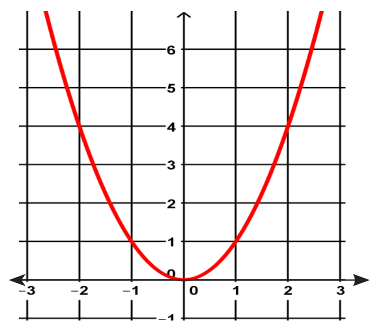
3. a. Draw the graph of the function $f(x) = x$.

   b. Draw the graph of the function $f(x) = x^2$.

   c. Draw the graph of the function $f(x) = x^3$.

4. Find the value of x or y at the given points from the following graphs.

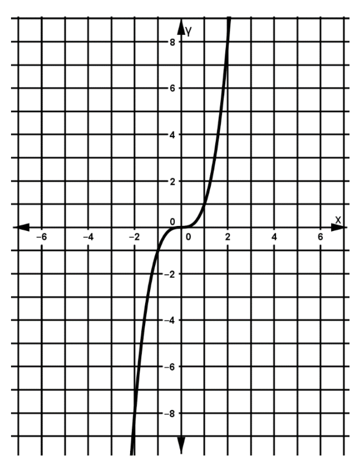




x	3	–1
y	....	....

x	....	....
y	4	1

5. The graph of a function $f(x)$ is given on right. From the graph, find the values of $f(0)$, $f(-1)$, $f(2)$.



Answer

1 - 3. Show to the teacher.

4. a. 3, –1 b. 2, 1

5. $f(0) = 0$, $f(-1) = -1$, $f(2) = 8$

1.2 Polynomials

The Greek mathematician Diophantus began the systematic study of various equations. The study of polynomials in symbolic form and the method of solving quadratic equations by completing the square were first introduced by the Islamic mathematician Al-Khwarizmi. Polynomials are used not only in mathematics but also in physics, computer science, and various other fields to solve problems.

Al-Khwarizmi
(780-850 C.E.)

1.2.1 Concept of Polynomial and Its Roots

Activity 1

Single Variable Polynomial

The figure on right shows a rectangle with length $2x + 3$ units and width $x + 2$ units.

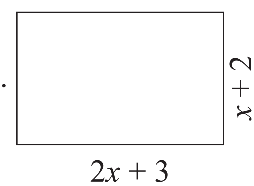
a. What is the area of the rectangle?

b. How many terms are there in the algebraic expression that represent the area?

c. What is the power of the variable in each term? List them.

d. Are all indices whole numbers?

e. What are the coefficients of the variable in each term?



Expressions like these in algebraic form cx^n are called polynomials in one variable x, where n is whole number and c is a constant. Polynomials are denoted by $p(x)$, $q(x)$, $r(x)$,...

For example: $p(x) = 5x^3 - 3x^2 + 2x - 6$, $q(x) = 3x - 2$ are polynomials, but $x^{-3} + 2x^2 - 5x$, $\frac{1}{x} + 5$ are not polynomials, because in x^{-3} the exponent -3 is not a whole number, and in $\frac{1}{x}$ the exponent -1 is not a whole number.

Example 1

Answer these questions:

a. How many terms are there in the polynomial $p(x) = x^3 + 3x^2 - 2x - 6$? What are the coefficients of x^3 and x? Write them.

b. Are $p(x) = \sqrt[4]{x^8} + 5x - 2$ and $q(x) = \sqrt[3]{x} + 5x - 2$ polynomials? Write with reason.

Solution

a. Here, $p(x) = x^3 + 3x^2 - 2x - 6$. There are four terms in this polynomial. Coefficients of x^3 and x are 1 and −2 respectively.

b. $p(x) = \sqrt[4]{x^8} + 5x - 2 = (x^8)^{\frac{1}{4}} + 5x - 2 = x^2 + 5x - 2$. Here, the power of the variable in each term is a whole number. So, $p(x)$ is a polynomial.

The second expression: $q(x) = \sqrt[3]{x} + 5x - 2 = x^{\frac{1}{3}} + 5x - 2$, $\sqrt[3]{x}$ can be written as $x^{\frac{1}{3}}$. Therefore, in $q(x) = x^{\frac{1}{3}} + 5x - 2$, the power of x is $\frac{1}{3}$, which is not a whole number. So, $q(x)$ is not a polynomial.

Standard Form of a Polynomial and Its Degree

A polynomial is said to be in standard form when the terms are arranged in ascending or descending order of the powers of the variable.

For example: $p(x) = 2x^4 - 5x^3 + 3x^2 + 5x - 4$ in standard form. Because the powers of all terms are arranged in descending order. Similarly, $q(x) = 2x^4 + 3x^2 - 4$ can be written in standard form by including missing terms with zero coefficients. i.e., $q(x) = 2x^4 + 0.x^3 + 3x^2 + 0.x - 4$.

The degree of a polynomial is the highest power of the variable in the polynomial. The coefficient of the highest degree term is called the leading coefficient.

Example 2

Write the polynomial $p(x) = 3x^5 - 6$ in standard form. What is its degree? What is the leading coefficient?

Solution

Given polynomial: $p(x) = 3x^5 - 6$

In standard form: $p(x) = 3x^5 + 0 \cdot x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x - 6$

Its degree is 5. The leading coefficient is 3.

Exercise 1.2 (A)

1. Define a polynomial with an example.
2. Write the coefficients of x^3 and x^2 in the polynomial $4x^5 + 2x^3 - 7x + 6$.
3. How many terms are in the polynomial $4x^6 - 7x + 6$? What is its degree?
4. Which of the following are polynomials and which are not? Write with reasons.

   a. $x^2 + \sqrt{5}$ b. $\sqrt{y} + 3$ c. $z + \frac{5}{z}$ d. $\sqrt{y} + y\sqrt{3}$

e. $x^3 - 7x^2 + 2x - 4$ f. $3x^2 - 11x + 6$ g. $4x^3 + \sqrt[3]{x}$

5. Write the degree of the following polynomials:

a. $2x^4 - 5x^2 - 1$ b. $x^{10} + 2x - 4 + 3x^3$

c. $x^5 + 2x^3 - 7x^7 + 6$ d. $-2x^3 + 5x^2 + 7x + x^5$

6. Write the following polynomials in standard form:

a. $2x^3 + 5x^2 + 7x + 9x^4$ b. $x^4 + 2x + 1 + 3x^3$

c. $x^5 + 2x^3 - 7x + 6$ d. $2x^4 + 7x^2 - 3$

Answer

1. Show to the teacher.
2. Coefficients of x^3 and x^2 are 2 and 0 respectively.
3. Number of terms = 3, Degree = 6 4. Show to the teacher.
5. a. 4 b. 10 c. 7 d. 5
6. a. $9x^4 + 2x^3 + 5x^2 + 7x$ b. $x^4 + 3x^3 + 0x^2 + 2x + 1$

c. $x^5 + 0x^4 + 2x^3 + 0.x^2 - 7x + 6$ d. $2x^4 + 0x^3 + 7x^2 + 0x - 3$

Root of the Polynomials

Activity 1

Let a polynomial be p$(x) = x^2 - 7x + 12$. Complete the table.

x	1	2	3	4	...
$p(x)$					

For which value of x does $p(x) = 0$?

In the given polynomial, the value of the variable for which the polynomial becomes zero is called the root of that polynomial. i.e. for $x = a$ if the polynomial $p(a) = 0$, then a is a root of polynomial $p(x)$

Alternatively,

If a is a root of $p(x)$, then $p(a) = 0$.

$p(a) = 0$

or, $a^2 - 7a + 12 = 0$

or, $(a - 4)(a - 3) = 0$

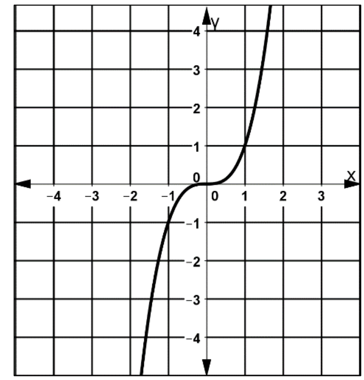
Either, $a - 3 = 0$ or, $a - 4 = 0$

i.e., $a = 3$ $a = 4$

Therefore, the roots of $p(x)$ are 3 and 4.

Example 1

The graph of $p(x) = x^3$ is given in the figure. Using the graph, find a root of $p(x)$.



Solution

In the given graph, $p(x)$ touches the X- axis at $x = 0$. Therefore, its root is 0.

That is, $p(0) = 0^3 = 0$.

Example 2

If one root of the polynomial $p(x) = 2x^2 – kx + 8$ is 4, what is the value of k? Find it.

Solution

Since one root of the given polynomial $p(x)$ is 4, therefore $p(4) = 0$.

$p(4) = 0$

Or, $2 \cdot 4^2 – k \cdot 4 + 8 = 0$

Or, $32 – 4k + 8 = 0$

Or, $40 – 4k = 0$

Or, $k = 10$

Therefore, the value of k is 10.

Exercise 1.2 (B)

1. What is meant by the root of a polynomial? Explain with an example. Also write its geometrical meaning.
2. a. If one root of the polynomial $p(x)$ is 2, at which point on the X-axis does its graph meet? Write it.
   b. If one root of the polynomial $p(x)$ is –1, at which point on the X-axis does its graph meet? Write it.
   c. If the roots of the polynomial $p(x)$ are 3 and –1, at which points on the X-axis does its graph meet? Write them.
3. Find the roots of the following polynomials:
   a. $p(x) = x + 3$ b. $p(x) = x^2 – 9$ c. $p(x) = x^2 – 1$
   d. $p(x) = 2x^2 – 7x + 6$ e. $p(x) = x^2 – 5x + 6$

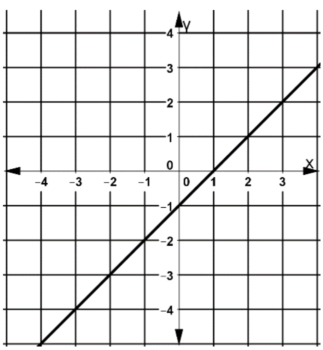
4. a. If one root of the polynomial $p(x) = kx + 3$ is 3, what is the value of k? Find it.

   b. If one root of the polynomial $p(x) = x^2 - kx + 6$ is 3, what is the value of k? Find it.

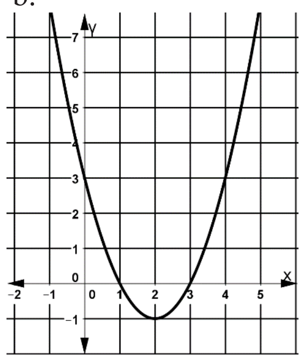
   c. If one root of the polynomial $p(x) = kx^2 - 25$ is 5, what is the value of k? Find it.

5. From the following graphs, find the root of the polynomial $p(x)$.

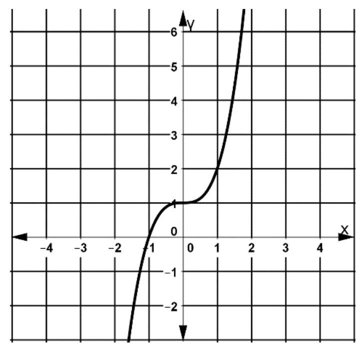
a.



b.



c.



Answer

1. Show to the teacher.
2. a. (2, 0) b. (–1, 0) c. (–1, 0) and (3, 0)
3. a. –3 b. 3 and –3 c. 1 and –1 d. 2 and $\frac{3}{2}$ e. 2 and 3
4. a. –1 b. 5 c. 1
5. a. 1 b. 1, 3 c. –1

1.2.2 Division of Polynomials

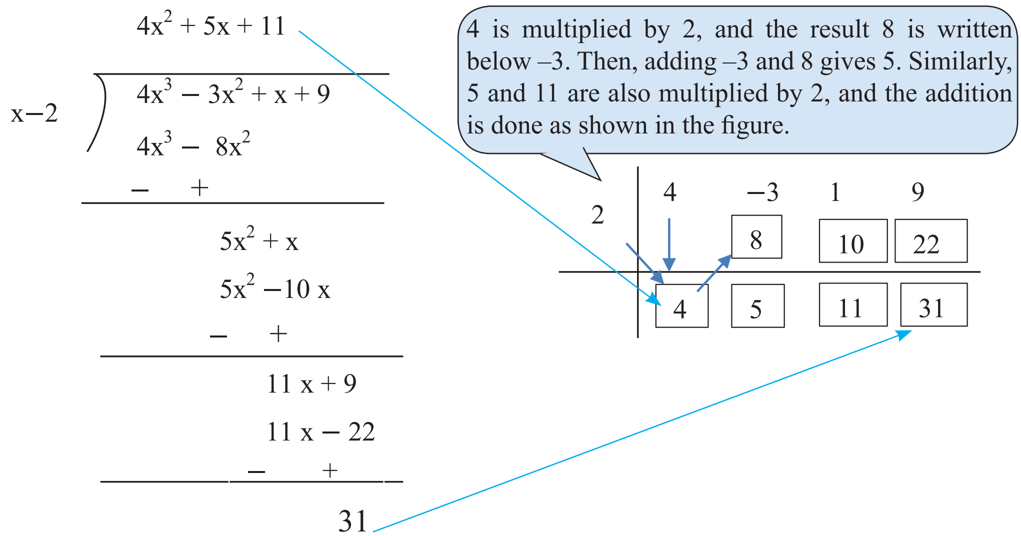
Synthetic Division

Synthetic division is a shortcut method for dividing a polynomial by a binomial. This method can be used to find factors of various polynomials and to find roots of polynomials.

To study this method, let's study the following division operation.

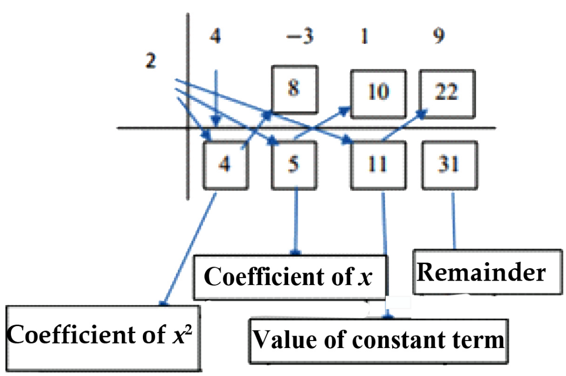
Divide the polynomial $4x^3 - 3x^2 + x + 9$ by $x - 2$ and find the quotient and remainder.

Here, divisor is $x - 2$ and dividend is $4x^3 - 3x^2 + x + 9$. Dividing by the conventional method:



The above division is done by shortcut method.

If the given polynomial is not in the standard form, it must be converted into the standard form. The given polynomial is in standard form. Now, our divisor is $x - 2$, so divide by 2. As shown in the figure, write the coefficients of the dividend polynomial and perform all the operations in one step.



Therefore, quotient = $4x^2 + 5x + 11$ and remainder = 31

Here, the degree of the divisor is 1 less than the degree of the dividend.

Note: The shortcut method of dividing a polynomial $p(x)$ by a binomial $(x - a)$ is called the synthetic division method. In this method, the degree of the quotient is one less than the degree of the dividend. This method is especially used to find factors, quotient and remainder of polynomials.

Example 1

Divide $x^3 - 3x^2 - 2x + 5$ by $x - 2$ and find the quotient and remainder.

Solution

Here the given polynomial is in standard form. Since the divisor is $x - 2$, we will divide by 2 in synthetic division. That is, comparing divisor $x - 2$ with $(x - a)$, we get $a = 2$.

Dividing using synthetic division:

2	1	−3	−2	5
	↓	2	−2	−8
	1	−1	−4	−3

−3 → Remainder

−4 → Value of constant term

−1 → Coefficient of x

1 → Coefficient of x^2

Quotient = $x^2 - x - 4$ Remainder = -3

Example 2

Using synthetic division, divide the polynomial $8x^3 - 1$ by $2x - 1$, and find the quotient and remainder.

Solution

Writing the given polynomial in standard form: $8x^3 + 0 \cdot x^2 + 0 \cdot x - 1$, the coefficients of the variable are 8, 0, 0, and -1 respectively. Since the divisor is $2x - 1$, comparing divisor $2x - 1$ with $(x - a)$, we get $2x - 1 = 2\left(x - \frac{1}{2}\right)$, $a = \frac{1}{2}$. Therefore, dividing by $\frac{1}{2}$:

According to synthetic division:

$\frac{1}{2}$	8	0	0	−1
	↓	4	2	1
	8	4	2	0

0 → Remainder

2 → Value of constant term

4 → Coefficient of x

8 → Coefficient of x^2

$\therefore$ Quotient = $\frac{1}{2}(8x^2 + 4x + 2) = 4x^2 + 2x + 1$ (Why?) Remainder = 0

Exercise 1.2 (C)

1. What is the degree of the divisor in synthetic division?
2. What is the difference between the degree of the quotient and the degree of the dividend in synthetic division?

   a. -1 b. 1   c. $n - 1$ d. n
3. Find the quotient and remainder using synthetic division when the following polynomials are divided by the given divisor:

   a. Dividend = $x^3 - 3x^2 + 4x - 3$, Divisor = $x - 1$

   b. Dividend = $2x^3 + 4x^2 - 5x + 6$, Divisor = $x - 2$

   c. Dividend = $2x^3 - 11x^2 - 19x - 10$, Divisor = $x - 5$

   d. Dividend = $y^3 - 3y^2 - 4y + 12$, Divisor = $y + 2$

   e. Dividend = $m^3 - 7m^2 + 7m + 15$, Divisor = $m + 1$

   (f) Dividend = $5x^2 - 6x - 9$, Divisor = $x - 2$
4. Find the quotient and remainder using synthetic division:

   a. Dividend = $2x^3 + 9x^2 + 7x - 6$, Divisor = $2x - 1$

   b. Dividend = $2x^3 + 3x^2 - 11x - 6$, Divisor = $2x + 1$

   c. Dividend = $6x^3 - 5x^2 - 3x + 2$, Divisor = $3x + 2$

   d. Dividend = $3x^3 - 13x^2 + 16$, Divisor = $3x - 4$

   e. Dividend = $6x^3 - 4 + 13x^2$, Divisor = $3x + 2$
5. a. If $2x^3 - 7x^2 + x + 10 = (x - 1) \cdot Q(x) + R$, find the value of $Q(x)$ and R.

   b. If $x^3 - 19x - 30 = (x + 2) \cdot Q(x) + R$, find the value of $Q(x)$ and R.

   c. If $x^3 - 21x - 20 = (x + 1) \cdot Q(x) + R$, find the value of $Q(x)$ and R.

Answer

1. 1   2. 1

3 a. Quotient = $x^2 - 2x + 2$ Remainder = -1 b. Quotient = $2x^2 + 8x + 11$ Remainder = 28

c. Quotient = $2x^2 - x - 24$ Remainder = -130 d. Quotient = $x^2 - 5x + 6$ Remainder = 0

e. Quotient = $x^2 - 8x + 15$ Remainder = 0   e. Quotient = $5x + 4$ Remainder = -1

4 a. Quotient = $x^2 - 5x + 6$ Remainder = 0   b. Quotient = $x^2 + x - 6$ Remainder = 0

c. Quotient = $2x^2 - 3x + 1$ Remainder = 0   d. Quotient = $x^2 - 3x - 4$ Remainder = 0

e. Quotient = $2x^2 + 3x - 2$ Remainder = 0

5 a. $Q(x) = 2x^2 - 5x - 4$, R = 6   b. $Q(x) = x^2 - 2x - 15$, R = 0

c. $Q(x) = x^2 - x - 20$, R = 0

1.3 Equation and Inequality

In our daily life, we frequently use expressions such as "at most two hours" when estimating the time required to travel somewhere, or terms like "at least fifty rupees" and "I can give at most one thousand rupees" when fixing the price of an item or requesting a discount from a shopkeeper. All these expressions are related to the concept of inequalities.

Look at the following examples:

– Mohan earns at most Rs. 90 per hour.

– Sarina earns at least Rs. 80 per hour.

If Mohan's hourly earning is denoted by x and Sarina's hourly earning by y, how can we represent their earnings in mathematical language? Let's think.

Representing Mohan and Sarina's earnings in mathematical language:

– Mohan's hourly earning $x \leq 90$

– Sarina's hourly earning $y \geq 80$

These two statements are called inequalities in mathematics.

Mathematical sentences using trichotomy symbols ($<$, $>$, $\leq$, $\geq$) are called inequalities. Inequalities involving algebraic expressions of first degree are called linear inequalities. For example: $x \leq 4$, $y > 10$, $x + 2y \geq 5$, etc.

The study of inequality is necessary to solve practical and theoretical problems in daily life as well as for further study in higher classes.

1.3.1 Linear Inequality of Single Variable and Its Graphical Representation

Activity 1

Objective: To represent the given statement in the form of an inequality.

Problem: How can the statement "It takes at least 2 hours to go from city A to city B" be represented as an inequality?

If a linear inequality has only one variable, then it is called a linear inequality with one variable. The different forms of linear inequalities in one variable are as follows:

Statement	Inequality
Value of x is less than a	$x < a$ or $a > x$
Value of x is greater than a	$x > a$ or $a < x$
Value of x is at least a, i.e., a or more than a	$x \geq a$ or $a \leq x$
Value of x is at most a, i.e., a or less than a	$a \geq x$ or $x \leq a$

Example 1

When a player runs 5 times faster than his/her competitor in a given time, the distance covered by him/her is 200 m or more than 200 m than his/her competitor.

a. Represent the above problem in an inequality.

b. Find the minimum distance covered by the player by solving the inequality.

Solution

Let, the distance covered by the player's competitor = x m, and distance covered by player = $5x$ m

a. Since the distance covered by him/her is 200 m or more than 200 m than his/her competitor,

$5x - x \geq 200$

b. Solving the above inequality,

Or, $5x - x \geq 200$

Or, $4x \geq 200$

Or, $x \geq \frac{200}{4}$ [Dividing both sides by 4]

Or, $x \geq 50$

Therefore, the minimum distance covered is 50 m.

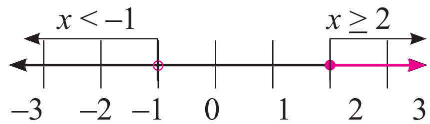
Points to remember while solving inequalities:

- Adding or subtracting the same number on both sides of the inequality does not change the inequality sign.
- Multiplying or dividing both sides of the inequality by the same positive number does not change the inequality sign.
- Multiplying or dividing both sides of the inequality by the same negative number changes the inequality sign.

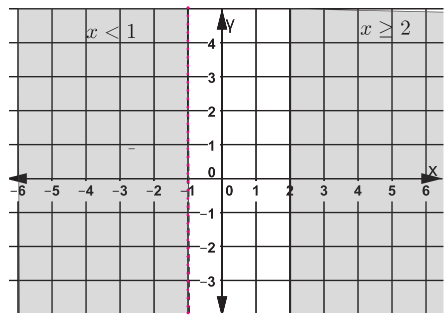
Activity 1

Discuss the method to represent $x \geq 2$ and $x < -1$ on a number line and graph.

For the first inequality $x \geq 2$,
Here, the value of x is 2 or greater than 2.



So, $x = 2$ is the boundary for the given inequality. There is solid dot at point $x = 2$ in numberline.



While in graph, first, plot the point x = 2. Since the inequality sign is "≥" (greater than or equal to), the boundary line should be a solid line. The value of x is 2 or more, so on the graph, shade the region to the right of $x = 2$ as shown in the figure.

For the second inequality x < –1,

Here, the value of x is less than –1.

So, $x = -1$ is the boundary line for the given inequality.

On the graph, first draw the boundary line for $x = -1$. Since the inequality sign is strictly less than (<), the boundary line should be a dotted line on the graph. The value of x is less than –1, so on the graph, shade the region to the left of $x = -1$ as shown in the figure.

The following steps are followed while representing the solution set of one or two linear inequalities on a graph:

– If the inequality has "≤" or "≥" sign, then the points on the boundary line are included in the solution set, so the boundary line should be a solid line.

– If the inequality has "<" or ">" sign, then the points on the boundary line are not included in the solution set, so the boundary line should be a dotted line.

Exercise 1.3 (A)

1. What do you mean by a linear inequality? Write with example.
2. Represent the following inequalities on a graph and show the solution set by shading:

a. $x \geq 0$	b. $x \leq 0$	c. $x \geq 3$	d. $x > 1$	e. $x \geq -2$
f. $x > -3$	g. $x < 2$	h. $y \geq 0$	i. $y \leq 0$	j. $y \geq 1$
k. $y > 3$	l. $y \geq -4$	m. $y > -5$	n. $y < 2$	

3. Solve the following inequalities and show the solution set on a number line:

a. $2x + 5 \leq 15$ b. $2x + 5 \geq 15$ c. $x + 3 > 7$

d. $5x \leq 20$ e. $\frac{7-x}{2} \geq 1$ f. $\frac{2+x}{3} < 1$

g. $(3x + 1) \geq \frac{2}{3}(4x - 3)$ h. $\frac{5}{4}(x + 1) \leq \frac{3}{2}(x - 1)$

i. $\frac{1}{3}(x + 2) \geq \frac{1}{5}(x + 6)$ j. $\frac{3x+1}{2} \leq \frac{2x-1}{3}$

k. $\frac{4x+1}{3} \geq \frac{3x-1}{2}$

4. Study the following statements, represent them as inequalities, and solve them:

a. Three times a number added to 4, is less than 13.

b. If 3 is subtracted from twice a number, the result is greater than or equal to 5.

c. If a motorcycle covers a distance with a speed of at least 40 km per hour and at most 60 km per hour, what should be the interval of time required to cover 120 km? Write it.

d. Sahabuddion obtained 72, 85, and 75 marks respectively in the first, second, and third exams in Optional Mathematics. If he wants his average marks to be at least 80, how many mark must he obtain in the fourth exam? Find it.

e. Lakpa has Rs. 500. He wants to buy half a dozen pens of one type costing Rs. 20 per pen, and another type of pen costing Rs. 40 per pen. How many of the second type of pen can he buy at most? Also, how much money will be left with him afterward?

f. Sanam asked to prepare two types of packets of prizes worth Rs 2000. The cost of first type of prize is Rs 300 per packet and the second type is Rs 500 per packet. If she needs 3 packets of first type of prize, how many packets of the second type of prize can she buy at most? How much money will be left?

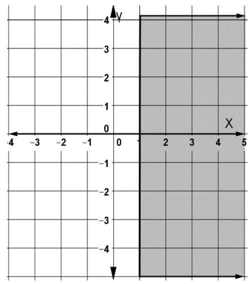
g. The base of a triangle is 5 cm. If its area lies between 20 sq. cm and 30 sq. cm, what is the interval of its height? Write it.

h. If the perimeter of a square is between 40 cm and 200 cm, find the possible length of its side satisfying this condition.

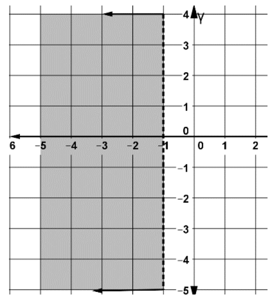
i. A scored 30 and 85 runs in the first two matches of T20 cricket series. How many runs must he/she score at least in the third match to have an average of at least 50 runs?

5. Represent the solution sets shown in the graphs below as linear inequalities:

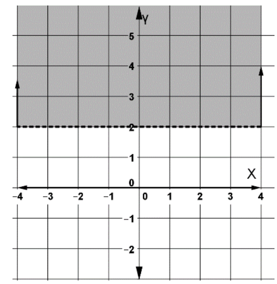
a.



b.



c.



Answer

1 - 2. Show to the teacher.

3. a. $x \leq 5$ b. $x \geq 5$ c. $x > 4$ d. $x \leq 4$ e. $x < 5$ f. $x < 1$
   g. $x \geq -9$ h. $x \geq 11$ i. $x \geq 4$ j. $x \leq -1$ k. $x \leq 5$ (Show graphs to the teacher.)

4. a. $x < 3$ b. $x \geq 4$ c. $2 \leq t \leq 3$ d. $x \geq 118$
   e. 9, 20 f. 2, 100 g. $2 < h < 3$ h. $10 \leq l \leq 50$ i. $x \geq 35$

5. a. $x \geq 1$ b. $x < -1$ c. $y > 2$

1.3.2 Linear Inequality of Two Variables and Graphical Representation

Activity 1

Samrat was given Rs. 30 by his mother to buy chocolates. There are two types of chocolates in the shop. The price of the first type of chocolate is Rs. 5 per piece, and the price of the second type is Rs. 10 per piece. Now, with the given amount of Rs. 30, how many chocolates of each type can Samrat buy at most? If he buys x pieces of the first type of chocolate and y pieces of the second type of chocolate, how can the total cost be expressed using an inequality? How can this inequality be represented on a graph? Discuss it.

The total amount given by the mother to Samrat is Rs. 30

Cost of x first type of chocolate at the rate of Rs. 5 each = Rs. $5x$

Cost of y second type of chocolate at the rate of Rs. 10 each = Rs. $10y$

Total cost of both types of chocolates ≤ 30

Now, representing the above condition as an inequality,

$5x + 10y \leq 30$

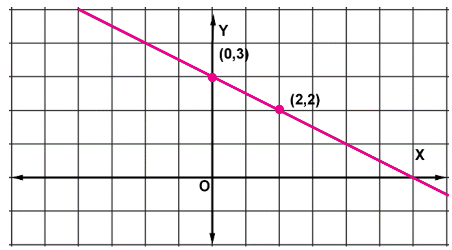
The above inequality has two variables x and y, hence such an inequality is called an inequality in two variables. To represent a two-variable inequality on a graph:

a. As in case of a one-variable inequality, we first need to find the boundary line of the given inequality. The boundary line for the above inequality is $5x + 10y = 30$.

b. To make the equation $5x + 10y = 30$ convenient, find the points (x, y) as follows:

x	0	2
y	3	2

Points: (0, 3) and (2, 2)



c. Plot the points (0, 3) and (2, 2) on the graph and join them making a straight line by scale. While drawing the line, pay attention to the inequality sign. Here, the inequality sign is "$\leq$", so the boundary line should be a solid line.

d. To find the solution region, test any point that does not lie on the boundary line. For this purpose, we can use the point (0, 0) in the given inequality. If the test point satisfies the inequality, shade the region that contains the test point. If it does not satisfy the inequality, shade the region on the opposite side of the test point. This indicates that for any point in the shaded region, the given inequality holds true or is valid.

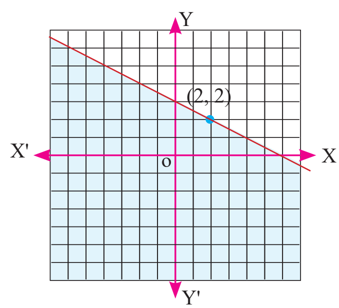
e. Testing the origin (0, 0):

$5x + 10y < 30$

Or, $5 \times 0 + 10 \times 0 < 30$

Or, $0 < 30$ (True)

So, shade the solution region containing the origin.



An inequality involving two variables is called a linear inequality in two variables. To represent such inequalities on a graph, first plot the equation of the boundary line. To identify the solution region, test a point that does not lie on the boundary line-commonly the point (0, 0). If the test point satisfies the inequality, shade the region containing that point; if it does not, shade the region on the opposite side.

Note: If line passes through (0, 0) then check other points.

Example 1

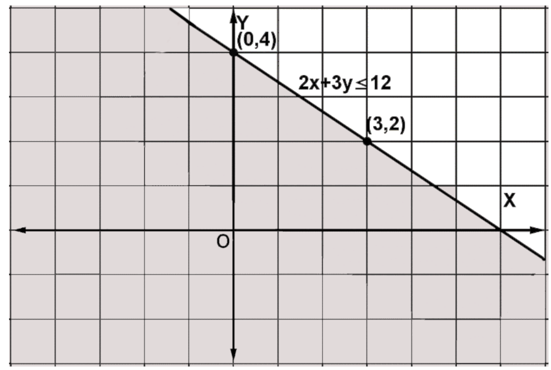
Represent $2x + 3y \leq 12$ on a graph.

Solution

Given inequality: $2x + 3y \leq 12$ Boundary line: $2x + 3y = 12$

Find the values of x and y that satisfy the above equation.

x	3	0
y	2	4



Plot the points (3, 2) and (0, 4) on the graph.

Test the origin (0, 0):

$2x + 3y < 12$

or, $2 \times 0 + 3 \times 0 < 12$

$0 < 12$ (True)

Exercise 1.3 (B)

1. What do you mean by a linear inequality in two variables? Write with example.
2. Represent the following inequalities on a graph by shading the solution region.

   a. $y \leq x + 5$ b. $y \geq 3x + 2$ c. $y \leq 2x - 3$ d. $2y \geq x - 4$

   e. $3x + 2y \geq 6$ f. $2x - y \leq 10$ g. $3x - 5y + 15 \geq 0$ h. $3x + 4y \leq 24$
3. Express the following statements as inequalities:

   a. Raman has Rs. 5000. He spends this amount to buy pants and T-shirts. If the price of each pant is Rs. 1000 and the price of each T-shirt is Rs. 500, express the possible numbers of pants and T-shirts he can buy as an inequality.

   b. Rita has Rs. 1000, with which she plans to buy two types of exercise books costing Rs. 50 and Rs. 80 each. Express the possible numbers of exercise books she can buy as an inequality.

   c. If buying 3 kg of apples and 2 kg of pomegranates costs less than Rs. 1,200, express the expenses as an inequality.

Answer

1 - 2. Show to the teacher. 3. a. $2x + y \leq 10$ b. $5x + 8y \leq 100$ c. $3x + 2y < 1200$

1.4 Number System

Discuss the following questions:

a. What is the difference between rational and irrational numbers?

b. Can you find the exact value of $\sqrt{16}$, $\sqrt[3]{27}$ and $\sqrt[3]{6}$?

c. Can $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{6}$ be called irrational numbers?

1.4.1 Surd and Its Operations

Let's study the following squares and cubes, and find their values:

$\sqrt{16}$ = ? $\sqrt{9}$ = ? $\sqrt[3]{64}$ = ? $\sqrt{4}$ = ? $\sqrt{3}$ = ? $\sqrt[3]{4}$ = ? $\sqrt{2}$ = ?

The values of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt[3]{4}$ cannot be expressed exactly. Therefore, all of these are surds. All these numbers are under the radical sign " $\sqrt{\ }$ " and the numbers associated with this sign, like 2, 3, and 4, are rational numbers. Their exact roots cannot be found. Such numbers are called surds. A number that contents a root sign is called an irrational number.

If the root of a number under a radical sign cannot be found as a whole number or a rational number, then such numbers are surds. Any surd $\sqrt[n]{a}$, where 'a' is a rational number and 'n' is a natural number, is called a surd of the n^{th} order. For example, the order of $\sqrt[3]{5}$ is 3; it is also called a third-order root. All surds are irrational numbers, but all irrational numbers may not be surds. For example, π, e, etc. Here, e is the Euler's number, where $e = 2.71828...$

Example 1

Write the order of the surds $\sqrt[4]{15}$ and $\sqrt[7]{12}$.

Solution

The orders of the given surds are 4 and 7 respectively.

Example 2

Express the surds $2\sqrt{3}$, $5\sqrt[3]{2}$ and $\sqrt[4]{5}$ as surds with the same order.

Solution

The orders of the given surds are 2, 3, and 4, respectively. Their L.C.M. is 12. Now, making the order of all surds 12:

$$2\sqrt{3} = 2\sqrt[2\times6]{3^6} = 2\sqrt[12]{729}$$

$$5\sqrt[3]{2} = 5\sqrt[3\times4]{2^4} = 5\sqrt[12]{16}$$

$$\sqrt[4]{5} = \sqrt[4\times3]{5^3} = \sqrt[12]{125}$$

Example 3

Arrange the surds $\sqrt{5}$, $\sqrt[3]{4}$ and $\sqrt[6]{27}$ in ascending order.

Solution

Here, the orders of the surds are 2, 3, and 6. The L.C.M. of 2, 3, 6 is 6. After making the order of all surds 6, they can be compared.

$$\sqrt{5} = \sqrt[2\times3]{5^3} = \sqrt[6]{125}$$

$$\sqrt[3]{4} = \sqrt[3\times2]{4^2} = \sqrt[6]{16}$$

$$\sqrt[6]{27} = \sqrt[6]{27}$$

Here, since $16 < 27 < 125$, therefore, $\sqrt[3]{4} < \sqrt[6]{27} < \sqrt{5}$

Operation on Surds

1. Only like surds can be added or subtracted. When adding or subtracting like surds, only the coefficients are added or subtracted. For example:

   a. $2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$ b. $2\sqrt{3} - 5\sqrt{3} = -2\sqrt{3}$

2. Let's look at the examples of multiplication and division of surds with the same order:

a. $7\sqrt{2}\times2\sqrt{5}$	b. $8\sqrt{10}\div2\sqrt{5}$
$=(7\times2)\sqrt{2\times5}$ $=14\sqrt{10}$	$=\frac{8\sqrt{10}}{2\sqrt{5}}$ $=\frac{8}{2}\times\sqrt{\frac{10}{5}}$ $=4\sqrt{2}$

From the examples, it can be concluded that when multiplying or dividing surds of the same order, the coefficients are multiplied/divided with coefficients and the surds are multiplied/divided with surds.

Let's look at an example with different orders:

$$\sqrt[3]{5}\div\sqrt{3}=\frac{\sqrt[3]{5}}{\sqrt{3}}=\frac{\sqrt[3\times2]{5^2}}{\sqrt[2\times3]{3^3}}=\frac{\sqrt[6]{25}}{\sqrt[6]{27}}=\sqrt[6]{\frac{25}{27}}$$

To multiply or divide surds with different orders, you must first convert them to surds with a same order. In the above example $\sqrt[3]{5}\div\sqrt{3}$, the orders of the surds are 3 and 2. The L.C.M. is 6, so they are expressed with the same order, 6.

Thought Provoking Question: After performing addition, subtraction, multiplication, and division of surds, is the result always a surd?

Laws of Radicals

For solving problems related to surds, some necessary laws are presented below:

1. $\sqrt[n]{a}=(a)^{\frac{1}{n}}$
2. $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$
3. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$
4. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$

Example 4

Simplify: $\sqrt{27}+\sqrt{75}-8\sqrt{3}$

Solution

Here, the given expression is:

$$\sqrt{27}+\sqrt{75}-8\sqrt{3}$$
$$=\sqrt{3\times3\times3}+\sqrt{3\times5\times5}-8\sqrt{3}$$
$$=3\sqrt{3}+5\sqrt{3}-8\sqrt{3}$$
$$=(3+5-8)\sqrt{3}$$
$$=0$$

Rationalization of Surd

The process of converting the denominator of a surd into a rational number is called rationalization.

For example: a. $\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$ Here, the surd in the denominator has been converted to a rational number.

b. $\frac{\sqrt{3}}{3\sqrt{2}}=\frac{\sqrt{3}}{3\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{6}}{3\times(\sqrt{2})^2}=\frac{\sqrt{6}}{6}$

The number which is multiplied with the given surd to get a rational number is called the rationalizing factor of that surd. For example, the rationalizing factor of $\sqrt{3}$ is $\sqrt{3}$ itself because $\sqrt{3}\times\sqrt{3}=3$, which is a rational number.

Similarly, the rationalizing factor of $(\sqrt{5}-\sqrt{3})$ is $(\sqrt{5}+\sqrt{3})$ because

$(\sqrt{5}-\sqrt{3})\times(\sqrt{5}+\sqrt{3})=(\sqrt{5})^2-(\sqrt{3})^2=5-3=2$, which is a rational number.

Therefore, $\sqrt{a}-\sqrt{b}$ and $\sqrt{a}+\sqrt{b}$ are rationalizing factors of each other.

Example 5

Simplify: $\frac{x-1}{\sqrt{x}+1}$

Solution

Here,

$$\frac{x-1}{\sqrt{x}+1}$$
$$=\frac{(\sqrt{x})^2-(1)^2}{\sqrt{x}+1}$$
$$=\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{(\sqrt{x}+1)}$$
$$=\sqrt{x}-1$$

Example 6

Simplify: $\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}} - \dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

Solution

Here,

$$\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}} - \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$= \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} - \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}} \times \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}-\sqrt{a}}$$ [Simplifying by rationalizing denominators]

$$= \frac{(\sqrt{x}+\sqrt{a})^2}{(\sqrt{x})^2-(\sqrt{a})^2} - \frac{(\sqrt{x}-\sqrt{a})^2}{(\sqrt{x})^2-(\sqrt{a})^2}$$

$$= \frac{x+2\sqrt{xa}+a-x+2\sqrt{xa}-a}{x-a}$$

$$= \frac{4\sqrt{xa}}{x-a}$$

Alternative Method

$$= \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}} - \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$= \frac{(\sqrt{x}+\sqrt{a})^2-(\sqrt{x}-\sqrt{a})^2}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}$$ [Taking LCM of denominator]

$$= \frac{(\sqrt{x})^2+2\sqrt{x}\sqrt{a}+(\sqrt{a})^2-\left((\sqrt{x})^2-2\sqrt{x}\sqrt{a}+(\sqrt{a})^2\right)}{(\sqrt{x})^2-(\sqrt{a})^2}$$

[$\because (a+b)^2 = a^2+2ab+b^2,\ a^2-b^2 = (a+b)(a-b)$

$$= \frac{x+2\sqrt{xa}+a-x+2\sqrt{xa}-a}{x-a}$$

$$= \frac{4\sqrt{xa}}{x-a}$$

Exercise 1.4 (A)

1. Define surd with an example.
2. Write the order of the following surds:

   a. $\sqrt{6}$ b. $\sqrt[3]{9}$ c. $3\sqrt[7]{6}$ d. $\sqrt[n]{a}$

3. a. For what value of m, x^3 and $\sqrt[3]{x^{m-1}}$ are equal?

   b. If y^4 and $\sqrt[4]{y^{2x}}$ are equal, what is the value of x?

4. Simplify:

   a. $\sqrt{3}\times\sqrt{5}$ b. $\sqrt{2}\times\sqrt{8}$ c. $\frac{\sqrt{15}}{\sqrt{5}}$ d. $\frac{\sqrt[3]{27}}{\sqrt{36}}$

   e. $\sqrt[3]{40}\times\sqrt[3]{25}$ f. $\frac{\sqrt[3]{64}}{\sqrt{16}}$

5. Simplify:

   a. $4\sqrt[3]{7}+2\sqrt[3]{7}-\sqrt[3]{7}$ b. $\frac{24}{\sqrt{3}}-\frac{18}{\sqrt{3}}$ c. $3\sqrt{3}+\frac{6}{\sqrt{3}}$ d. $\sqrt[4]{16807}$

   e. $\sqrt[3]{250}$ f. $3\sqrt{2}+\sqrt[4]{2500}+\sqrt[4]{64}+6\sqrt{8}$

   g. $\sqrt{50}+\sqrt{18}-8\sqrt{2}$ h. $4\sqrt[3]{250}-8\sqrt[3]{128}+4\sqrt[3]{54}$

   i. $4\sqrt[3]{24}-11\sqrt[3]{192}+6\sqrt[3]{648}$ j. $\sqrt{125}-\sqrt{45}+\sqrt{5}$

6. Simplify:

   a. $\sqrt{5}\times\sqrt{2}\times\sqrt{10}$ b. $\sqrt[3]{54}\times\sqrt[3]{4}$

   c. $6\sqrt[3]{4}\times 7\sqrt[3]{12}$ d. $\sqrt{40}\times\sqrt{18}$

   e. $(2\sqrt{5}+3\sqrt{2})\times(3\sqrt{5}-4\sqrt{2})$ f. $(\sqrt{2}+4\sqrt{3})\times(\sqrt{12}-2\sqrt{3})$

   g. $(3-2\sqrt{3})^2$ h. $(5+2\sqrt{2})^2$

   i. $(7\sqrt{5}-2\sqrt{7})^2$ j. $(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})$

   k. $(\sqrt{5}+\sqrt{3})\times(\sqrt{5}-\sqrt{3})$

   l. $\frac{\sqrt[3]{128}-\sqrt[3]{16}}{2\sqrt[3]{2}}$ m. $\frac{\sqrt{12}+\sqrt{27}}{\sqrt{3}}$ n. $\frac{6\sqrt{45}-2\sqrt{80}-3\sqrt{20}}{2\sqrt{180}}$

7. Rationalize the denominators and simplify:

   a. $\frac{12}{\sqrt{7}-\sqrt{5}}$ b. $\frac{2+\sqrt{7}}{2-\sqrt{7}}$ c. $\frac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}-\sqrt{18}}$ d. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

e. $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ f. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

g. $\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$ h. $\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}+\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

i. $\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}$ j. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$

Answer

1. Show to the teacher. 2. a. 2 b. 3 c. 7 d. n

3 a. 10 b. 8

4 a. $\sqrt{15}$ b. 4 c. $\sqrt{3}$ d. $\frac{1}{2}$ e. 10 f. 1

5 a. $5\sqrt[3]{7}$ b. $2\sqrt{3}$ c. $5\sqrt{3}$ d. $7\sqrt[4]{7}$

e. $5\sqrt[3]{2}$ f. $7\sqrt[4]{4}+15\sqrt{2}$ g. 0 h. 0 i. 0 j. $3\sqrt{5}$

6 a. 10 b. 6 c. $84\sqrt[3]{6}$ d. $12\sqrt{5}$ e. $6+\sqrt{10}$ f. 0 g. $3(7-4\sqrt{3})$

h. $33+20\sqrt{2}$ i. $27-28\sqrt{35}$ j. a – b k. 2 *l*. 1 m. 5 n. $\frac{1}{3}$

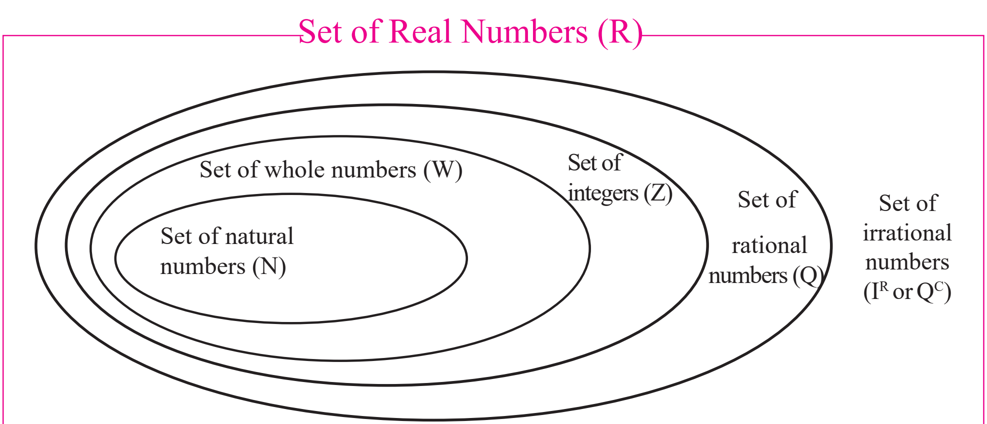
7 a. $6(7+\sqrt{5})$ b. $\frac{11+4\sqrt{7}}{-3}$ c. $\frac{54+\sqrt{6}}{20}$ d. $2+\sqrt{3}$ e. 8 f. 10

g. $2x^2-2x\sqrt{x^2-1}-1$ h. $\frac{2(x+a)}{x-a}$ i. $2x$ j. $4x\sqrt{x^2-1}$

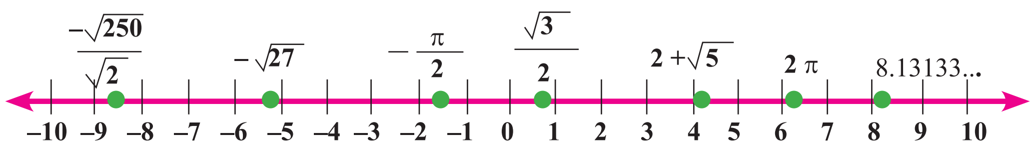
1.4.2 Real Numbers

Activity 1

Study the following diagram, and find the sets of different types of numbers and their subsets.



The union of the sets of rational and irrational numbers is the set of real numbers. The real numbers can also be represented on the real number line as follows:



Thought Provoking Question: How many real numbers are there between any two real numbers?

There are infinite real numbers between any two real numbers. For example, between the two real numbers 1 and 2, there are infinite real numbers such as 1.01, 1.001, 1.4, 1.0011, 1.00005, ... Thus, there are infinitely many real numbers between any two real numbers.

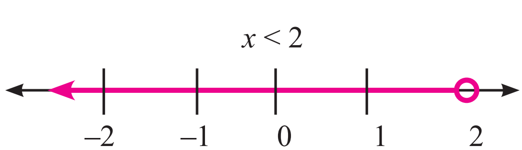
The union of the sets of rational and irrational numbers is the set of real numbers. There are infinite real numbers between any two distinct real numbers. To visualize real numbers, we represent them on the number line.

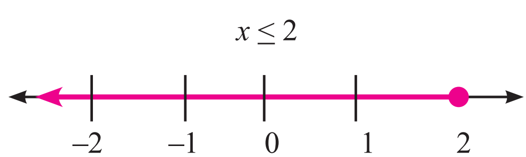
Concept of Interval

Activity 1

Problem: x is a real number and $x < 2$, $x \leq 2, x > 2, x \geq 2, -1 < x < 2$ and $-1 \leq x \leq 2$. Write their meaning and show them on the number line.

$x < 2$ means the value of x is all real numbers less than 2, excluding 2. Since the real numbers less than 2 are infinite, this set has infinite members.

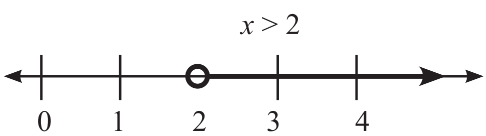


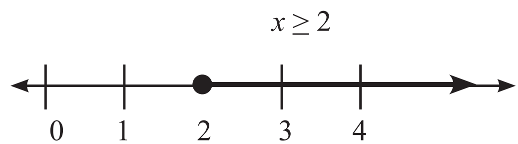


Similarly, $x \leq 2$ means the value of x is all real numbers less than or equal to 2, including 2 and the infinite real numbers less than 2. This set also has infinite members. This can be represented on the number line as follows:

Similarly, $x > 2$ and $x \geq 2$:

This can be represented on the number line as follows:



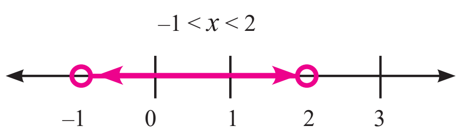


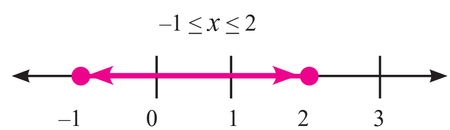
- $x > 2$ means the set of all real numbers greater than 2, excluding 2.
- $x \geq 2$ means the set of all real numbers greater than or equal to 2, including 2.

Let's take another example: $-1 < x < 2$ and $-1 \leq x \leq 2$.

- $-1 < x < 2$ means the value of x is all real numbers between -1 and 2, excluding –1 and 2. This set contains infinite real numbers.
- Similarly, $-1 \leq x \leq 2$ means the value of x is all real numbers between –1 and 2, including –1 and 2. This set also contains infinite real numbers.

This can be represented on the number line as follows:





From the above example, we can say that the numbers between two real numbers –1 and 2 is denoted by $-1 < x < 2$ and $-1 \leq x \leq 2$. Here, – and 2 are called the endpoints of the interval. In the first case, –1 and 2 are not included, while in the second case, both are included. $x < 2$, $x > 2$, $x \leq 2$, and $x \geq 2$ are not called intervals because they do not have two definite endpoints.

> The set of all real numbers between any two real numbers taken as endpoints, without any gap, is called an interval. Such intervals contain infinite real numbers. Such intervals, the end point may or may not be included. The interval between two real numbers a and b (where $a < b$) can be denoted by $a \leq x \leq b$ if the endpoints are included, and by $a < x < b$ if they are not included.

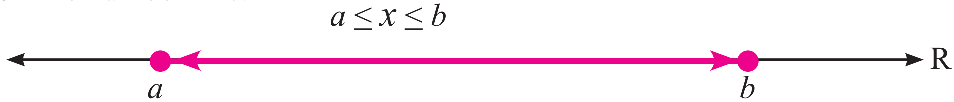
Intervals are classified as follows:

a. Closed Interval

If a and b are two real numbers where a is smaller than b, then the set of all real numbers between a and b including a and b is called a closed interval from a to b. It is denoted by $[a, b]$.

Mathematically, $[a, b] = \{x : a \leq x \leq b, x \in R\}$

On the number line:



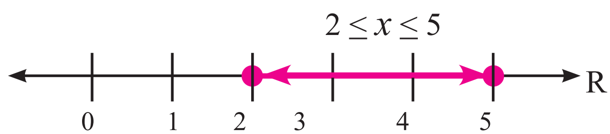
Example 1

What do you mean by closed interval from 2 to 5? Write it in notation and represent on the number line.

Solution

The closed interval from 2 to 5 means $2 \leq x \leq 5$, where 2 and 5 are included. It is denoted by $[2, 5]$.

Mathematically, $[2, 5] = \{x : 2 \leq x \leq 5, x \in R \}$



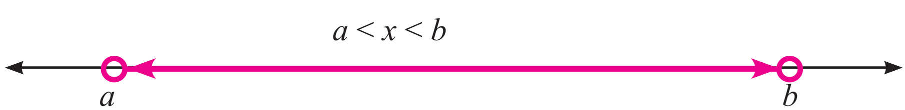
On the number line:

b. Open Interval

If a and b are two real numbers where a is smaller than b, then the set of all real numbers excluding a and b is called an open interval from a to b. It is denoted by (a, b).

Mathematically, $(a, b) = \{ x : a < x < b, x \in R \}$

On the number line:



Example 2

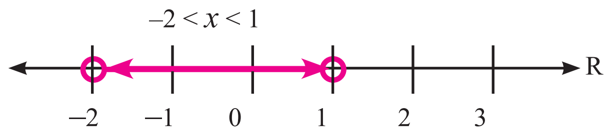
What is meant by the open interval from –2 to 1? Write it in set notation and represent it on the number line.

Solution

The open interval from –2 to 1 means the set of all real numbers between –2 and 1, excluding –2 and 1. It is denoted by (–2, 1).

Mathematically, $(-2, 1) = \{ x : -2 < x < 1, x \in R \}$

On the number line:

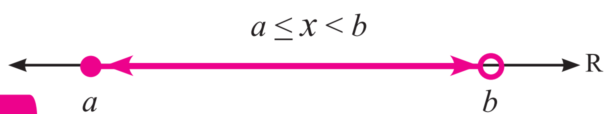


c. Left Closed and Right Opened Interval

Let the two definite real numbers of the interval be a and b where a is less than b. The left-closed right-open interval from a to b is the set of all real numbers x from a to b, including a but excluding b. It is denoted by $[a, b)$.

Mathematically, $[a, b) = \{ x : a \leq x < b, x \in R \}$

On the number line:



Example 3

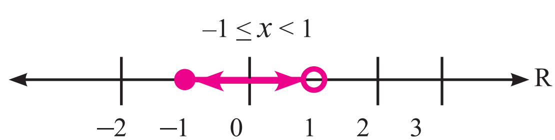
What is meant by the interval [–1, 1)? Write it in set notation and represent it on the number line.

Solution

The interval [–1, 1) means the set of all real numbers from –1 to 1, including –1 but excluding 1. It is denoted by [–1, 1).

Mathematically, $[-1, 1) = \{x : -1 \leq x < 1, x \in R \}$

On the number line:

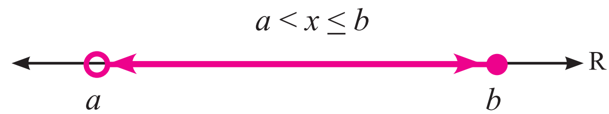


d. Left Open and Right Closed Interval

Let the two definite real numbers of the interval be a and b where a is less than b. The left-open right-closed interval from a to b is the set of all real numbers greater than a and less than or equal to b, excluding a but including b. It is denoted by $(a, b]$.

Mathematically, $(a, b] = \{ x : a < x \leq b, x \in R \}$

On the number line:



Example 4

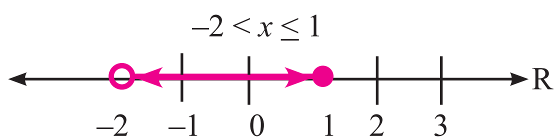
What do you mean by the interval (–2, 1]? Write it in notation and represent on the number line.

Solution

The interval (–2, 1] means the set of all numbers greater than –2 and less than or equal to 1, excluding –2 but including 1.

Mathematically, $(-2, 1] = \{x : -2 < x \leq 1, x \in R \}$

On the number line:



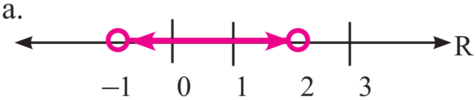
Exercise 1.4 (B)

1. What do you mean by a closed interval? Write with an example.
2. What do you mean by open interval? Write with an example.
3. Which of the intervals are right opened-left closed interval?

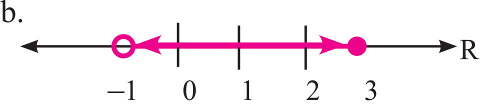
   a. $[a, b]$ b. $(a, b]$ c. $[a, b)$ d. (a, b)
4. Show the following intervals on the number line with their meaning:

   a. [1, 3] b. (–2, 4) c. (–3, 0) d. (0, 5)
5. Write the intervals shown in the number lines below:

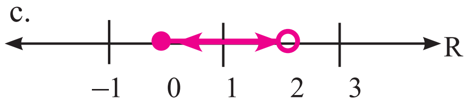
a.



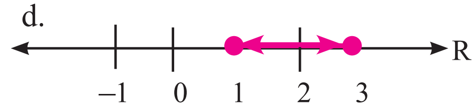
b.



c.



d.



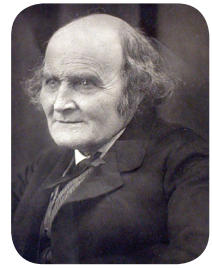
Answer

1 - 2. Show to the teacher. 3. c. $[a, b)$ 4. Show to the teacher.

5. a. (–1, 2) b. [–1, 3] c. [0, 2] d. [1, 3]

1.5 Matrix

The term 'Matrix' was first introduced by Sylvester in 1850. History shows that in 1858, Cayley began the English alphabet, such as A, B, ..., to denote matrices.


Cayley

Matrices are very essential in various branches of mathematics. They are used in solving daily life problems and in advanced fields like computer graphics, statistics, economics, linear algebra, physics, algebra, chemistry theory, cryptography, geometry, and various other fields of science.

1.5.1 Introduction to Matrix

Activity 1

Look at the following table:

The price of a sack of rice in different shops are given.

Shop/Rice	Basmati	Jira masino	Sona mansuli
A	Rs. 2200	Rs. 2000	Rs. 2450
B	Rs. 2210	Rs. 2050	Rs. 2500
C	Rs. 2190	Rs. 2010	Rs. 2480

The above table can be represented by writing the names of the shop and price in rows (horizontally) and the types of rice in columns (vertically) as follows:

$$\begin{array}{lc} & \\ \textbf{Row} \rightarrow \textbf{1} & A \\ \textbf{Row} \rightarrow \textbf{2} & B \\ \textbf{Row} \rightarrow \textbf{3} & C \end{array} \quad \begin{array}{ccc} \text{Basmati} & \text{Jira masino} & \text{Sona mansuli} \\ \left[\begin{array}{c} 2200 \\ 2210 \\ 2190 \end{array}\right. & \begin{array}{c} 2000 \\ 2050 \\ 2010 \end{array} & \left.\begin{array}{c} 2450 \\ 2500 \\ 2480 \end{array}\right] \\ \uparrow & \uparrow & \uparrow \\ \text{Column 1} & \text{Column 2} & \text{Column 3} \end{array}$$

The rectangular array of numbers enclosed in brackets '[ ]' above is called a matrix. It can be represented by capital letters like A, B, C, etc.

$$R = \begin{bmatrix} 2200 & 2000 & 2450 \\ 2210 & 2050 & 2500 \\ 2190 & 2010 & 2480 \end{bmatrix}$$

The values in the matrix R are called elements.

A rectangular arrangement of numbers in the form of rows and columns is called a matrix. A matrix is written inside brackets [ ] or ( ). Generally, matrices are denoted by capital letters such as A, B, C, etc. The elements of a matrix are denoted by small letters such as *a*, *b*, *c*, etc. A matrix is only a method of representing data; it does not have its own numerical or quantitative value.

Example 1

The grade points obtained by three students in three subjects are given in the table below. Write the information in the matrix

Name of students	Subjects		
	Mathematics	Science	English
Sanam	3.8	3.6	3.2
Rahaman	3.6	3.2	3.8
Pemba	3.8	3.6	3.2

Solution

The data in the table above can be presented in matrix as follows:

$$A = \begin{bmatrix} 3.8 & 3.6 & 3.2 \\ 3.6 & 3.2 & 3.8 \\ 3.8 & 3.6 & 3.2 \end{bmatrix}$$

Order of Matrix

In the given the matrix $X = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$, how many rows and columns are there? Write its order using number of rows and columns. How many elements are in the matrix X? Is the product of rows and columns equal to the number of element? Investigate it.

The order of matrix X is written as (number of rows) × (number of columns), so it is 3 × 2. This is read as '3 by 2 matrix' and written as $X_{3 \times 2}$.

The representation of a matrix as (number of rows) × (number of columns) is called the order of the matrix. So, order of a matrix = Number of Rows × Number of Columns. For a matrix $A_{m \times n}$ the first number 'm' represents the number of rows and the second number 'n' represents the number of columns.

Total number of elements in a matrix = Number of Rows × Number of Columns.

Example 2

Find the order of the following matrices:

a. $A = [1 \;\; 0 \;\; 3]$

b. $P = \begin{bmatrix} 1 & -2 & 3 \\ -4 & -3 & 0 \end{bmatrix}$

c. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d. $V = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

Solution

a. Order of matrix A = 1 × 3

b. Order of matrix P = 2 × 3

c. Order of matrix I = 2 × 2

d. Order of matrix V = 3 × 3

Example 3

If the total number of elements in a matrix is 10, what can be the possible orders of that matrix? Write them.

Solution

Here, total number of elements in the matrix = 10. For a matrix of order $m \times n$, the total number of elements is $m \times n$.

Therefore, finding the pairs of natural numbers whose product is 10:

(1, 10), (10, 1), (2, 5), (5, 2)

Possible orders = 1 × 10, 10 × 1, 5 × 2 and 2 × 5

Positions of Elements of Matrix

Study the following examples:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

In the above example,

a_{11} = element of 1st row and 1st column = 1

a_{12} = element of 1st row and 2nd column = –2

a_{13} = element of 1st row and 3rd column = 3

a_{21} = element of 2nd row and 1st column = –4

a_{22} = element of 2nd row and 2nd column = – 3

a_{23} = element of 2nd row and 3rd column = 0

Thus, the elements can be easily identified from the position of the elements in matrix A. Each element of a matrix has a unique position. The position of each of these elements can be defined by the row number and column number. Any element of a matrix A is denoted by a_{ij} where, i represents the row number and j represents the column number.

> The elements of a matrix are written in the form a_{mn}, where m is the row number and n is the column number. Using the positions of elements, matrices of different orders can be easily constructed.
>
> Note: Based on the positions of elements, a matrix can be represented as follows:
>
> $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$
>
> Thus, the above matrix is denoted by $A = [a_{ij}]_{m \times n}$ where $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$

Example 4

A matrix $Q = \begin{bmatrix} 0 & -1 & 5 & 3 \\ 2 & 4 & 5 & 0 \\ 1 & -3 & 8 & 0 \end{bmatrix}$ based on this, answer the following questions:

a. Write the order of matrix Q.

b. Write the values of a_{21}, a_{14}, a_{33} and a_{24} c. What is the value of $a_{21} + a_{34} - a_{23}$? Write it.

Solution: Here,

a. Order of matrix Q is 3 × 4.

b. Here, $Q = \begin{bmatrix} 0 & -1 & 5 & 3 \\ 2 & 4 & 5 & 0 \\ 1 & -3 & 8 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

a_{21} = 2nd row and 1st column = 2 a_{14} = 1st row and 4th column = 3

a_{33} = 3rd row and 3rd column = 8 a_{24} = 2nd row and 4th column = 0

c. $a_{21} + a_{34} - a_{23}$

$= 2 + 0 - 5$

$= -3$

Example 5

Construct a matrix of order 3 × 4 whose elements are given by $a_{ij} = 2i + j$.

Solution

Suppose, $A = [a_{ij}]_{3 \times 4}$ then $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Here, $a_{ij} = 2i + j$

Therefore,

$a_{11} = 2 \times 1 + 1 = 3$

$a_{12} = 2 \times 1 + 2 = 4$

$a_{13} = 2 \times 1 + 3 = 5$

$a_{14} = 2 \times 1 + 4 = 6$

$a_{21} = 2 \times 2 + 1 = 5$

$a_{22} = 2 \times 2 + 2 = 6$

$a_{23} = 2 \times 2 + 3 = 7$

$a_{24} = 2 \times 2 + 4 = 8$

$a_{31} = 2 \times 3 + 1 = 7$

$a_{32} = 2 \times 3 + 2 = 8$

$a_{33} = 2 \times 3 + 3 = 9$

$a_{34} = 2 \times 3 + 4 = 10$

Required matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 \end{bmatrix}$

Exercise 1.5 (A)

1. Answer the following questions:

   a. Define a matrix with an example. Also write the symbol to denote it.

   b. What do you mean by the order of a matrix? Explain with an example.

   c. In the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ what is the value of a_{22}? Write it.

   d. In any matrix A, which position does a_{23} represent?

      (i) Second row third column (ii) Second column third row

      (iii) Third row second column (iv) Third column second row

   e. If the order of a matrix is 3 × 2, then which of the following is the total number of elements in the matrix?

      (i) 5 (ii) 1 (iii) 6 (iv) 9

2. Find the order of the following matrices with notation:

a. $P = [1 \quad 4 \quad 2]$ b. $X = \begin{bmatrix} 0 & 4 \\ -7 & 0 \\ 2 & -3 \\ 1 & -5 \end{bmatrix}$ c. $M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -3 & -2 \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, answer the following:

a. What is the element a_{32}?

b. What is the value of $(a_{32} + a_{21})^2$? Write it.

c. What is the value of $a_{31} - a_{22} + a_{13}$? Write it.

4. Construct the given matrices:

a. Matrix $A = [a_{ij}]_{2\times3}$ where $a_{ij} = 3i + 2j$

b. Matrix $B = [b_{ij}]_{3\times2}$ where $b_{ij} = 2i^2 - j$

c. Matrix $X = [x_{ij}]_{3\times3}$ where $x_{ij} = |2i - j|$

d. Matrix $M = [m_{ij}]_{2\times4}$ where $m_{ij} = \dfrac{(i-j)^2}{2}$

e. Matrix $P = [p_{ij}]_{2\times3}$ where $p_{ij} = (i \times j)^2$

5. Write the grade point of three students in three subjects. Represent this information in a matrix and find the order and elements by discussing with friends.

Answer

1. a. and b. Show to the teacher. c. 0 d. Second row Third column e. 6

2. a. Order of P = 1 × 3 , $P = [p_{ij}]_{1\times3}$ b. Order of X = 4 × 2 , $X = [x_{ij}]_{4\times2}$
c. Order of M = 3 × 2, $M = [m_{ij}]_{3\times2}$ 3. a. 0 b. 4 c. 4

4. a. $A = \begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \end{bmatrix}$ b. $B = \begin{bmatrix} 1 & 0 \\ 7 & 6 \\ 17 & 16 \end{bmatrix}$ c. $X = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$

d. $M = \begin{bmatrix} 0 & \frac{1}{2} & 2 & \frac{9}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 2 \end{bmatrix}$ e. $P = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \end{bmatrix}$

5. Show to the teacher.

Types of Matrices

Look at the following examples:

a. Row matrix	b. Column matrix	c. Rectangular matrix	d. Square matrix
$A = [1 \quad 2 \quad 3]$	$B = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$	$C = \begin{bmatrix} 7 & 9 \\ -8 & 5 \\ 3 & 6 \end{bmatrix}$	$D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

In the above examples:

– The first matrix A has only one row. Therefore, it is called a row matrix.

– The second matrix B has only one column. Therefore, it is called a column matrix.

– The third matrix C has more than one row and column and the numbers of rows and columns are not equal. Such a matrix is called a rectangular matrix.

– The fourth matrix D has an equal number of rows and columns. Such a matrix is called a square matrix.

Thought Provoking Question: Can all matrices be called rectangular matrices?

e. Zero or Null Matrix

What are the orders of the matrices $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$?

What are their elements like?

All the given matrices have elements as zero. Therefore, regardless of the order, since all elements are zero, the above matrices are called zero matrices.

A matrix in which all elements are zero is called a zero matrix. Zero matrices are denoted by $O_{m \times n}$.

f. Diagonal Matrix

Answer the following questions thoughtfully:

In square matrices $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$, which elements are on the main diagonals?

Which elements are there other than the main diagonal? What is such a matrix called?

In the first square matrix M above, the main diagonal has 1 and 2, and all other elements are zero. Similarly, in the square matrix N, the main diagonal has 1, 5, and –4, and all other elements are zero. In both square matrices, all elements except the main diagonal are zero, so such matrices are called diagonal matrices.

Thought Provoking Question: Can square matrices $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ also be called diagonal matrices?

A square matrix in which all elements except the main diagonal are zero is called a diagonal matrix. For a diagonal matrix $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0$ for $i \neq j$.

g. Scalar Matrix

In the square matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ what are the elements on the main diagonal?

Are they all equal? What are the elements other than the main diagonal?

In both of the above square matrices, the main diagonal elements are the same, and all elements except the main diagonal are zero. Such square matrices are called scalar matrices.

A diagonal matrix in which all the main diagonal elements are the same (equal) is called a scalar matrix. The above matrices are scalar matrices. Generally, for a diagonal matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = k$ for $i = j$.

h. Unit or Identity Matrix

In the square matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ what are the elements on the main diagonal?

Are they all equal? How many are there? What are the elements other than the main diagonal?

In both of the above square matrices, all main diagonal elements are 1, and all elements except the main diagonal are zero. Such square matrices are called unit or identity matrices.

A scalar matrix in which all the main diagonal elements are 1 is called a unit or identity matrix. The above matrices are scalar matrices. Generally, for a diagonal matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$. Identity matrices are denoted by I. For example, the 2 × 2 identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thought Provoking Question: Can all identity matrices be called scalar matrices?

i. Triangular Matrix

Study and discuss the following matrices:

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 8 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 8 & -3 & 6 \end{bmatrix}$$

Explore the special characteristics of these matrices.

Here, in the first two matrices, the elements below the main diagonal are zero. Since all elements below the main diagonal are zero, the first two square matrices are called upper triangular matrices.

Similarly, in the last two matrices, all elements above the main diagonal are zero. Since all elements above the main diagonal are zero, the last two square matrices are called lower triangular matrices.

Thus, all the above matrices are called triangular matrices.

A square matrix in which all elements above or below the main diagonal are zero is called a triangular matrix. It is of two types: upper triangular matrix and lower triangular matrix. For a square matrix A, if it is a triangular matrix, then for $A = [a_{ij}]_{n\times n}$, $a_{ij} = 0$ for $i > j$ or $i < j$.

j. Symmetric Matrix

In a square matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 8 \\ 2 & 8 & 6 \end{bmatrix}$, write all the elements in the rows into columns and the elements in the columns into rows. Did you find any difference? Is it the same matrix?

In the matrix A, when rows and columns are interchanged, the same matrix is obtained. Because of this property, matrix A is called a symmetric matrix.

In a square matrix, when the columns are interchanged with rows and the rows are interchanged with columns (i.e., the matrix is transposed) and no change occurs in the matrix, then such a matrix is called a symmetric matrix. If matrix A is a symmetric matrix, then for $A = [a_{ij}]_{n\times n}$, $a_{ij} = a_{ji}$ for $i \neq j$.

Equal Matrices

Square matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ are given. Here a_{ij} and b_{ij} are called corresponding elements.

For example, a_{12} and b_{12} are corresponding elements. So, to check if any two matrices are equal or not, one must check the order and the corresponding elements.

For example: $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 9 \\ -8 & 5 \\ 3 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 9 \\ -8 & 5 \\ 3 & 6 \end{bmatrix}$

Here, matrices A and C have the same order and the corresponding elements are also equal. Therefore, these two matrices are called equal matrices. This is denoted by A = C. Similarly, matrices B and D have the same order and the corresponding elements are also equal. Therefore, these two matrices are called equal matrices. This is denoted by B = D.

But matrices A and B do not have the same order and the corresponding elements are also not equal. Therefore, these two matrices are not called equal matrices. This is denoted by A ≠ B. Similarly, A ≠ D.

Matrices that have the same order and equal corresponding elements are called equal matrices. Similarly, if two matrices are equal, then their orders and corresponding elements are also equal.

For example: If $\begin{bmatrix} z & x \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$ then $a = 1$, $b = 5$, $x = 3$ and $z = 2$.

Example 1

Define a diagonal matrix with an example.

Solution

A square matrix in which all elements except the main diagonal elements are zero, then such a matrix is called a diagonal matrix. For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Example 2

If $\begin{bmatrix} x & 4 \\ 5 & y \end{bmatrix} = \begin{bmatrix} 2 & a^2 \\ 10b & 1 \end{bmatrix}$ then find the values of x, y, a and b.

Solution

Given: $\begin{bmatrix} x & 4 \\ 5 & y \end{bmatrix} = \begin{bmatrix} 2 & a^2 \\ 10b & 1 \end{bmatrix}$

Since the two matrices are equal, their corresponding elements are also equal. Therefore, equating corresponding elements, we get,

$x = 2$	$y = 1$	$a^2 = 4$ Or, $a = \pm\sqrt{4}$ Or, $a = \pm 2$	$10b = 5$ Or, $b = \frac{5}{10}$ Or, $b = \frac{1}{2}$

$\therefore x = 2, y = 1, a = \pm 2, b = \frac{1}{2}$

Example 3

For what values of x and y are the matrices $\begin{bmatrix} 2x+y & 4 \\ 5 & 8 \end{bmatrix}$ and $\begin{bmatrix} 6 & 4 \\ 5 & x+y \end{bmatrix}$ equal? Find it.

Solution

Suppose, $\begin{bmatrix} 2x+y & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 5 & x+y \end{bmatrix}$

Equating corresponding elements,

$2x + y = 6 \ldots$ (i)

$x + y = 8 \ldots$ (ii)

Subtracting the above two equations,

$$\begin{array}{rcl} 2x+y & = & 6 \\ \underline{-x+-y} & = & \underline{-8} \\ x & = & -2 \end{array}$$

Substituting the value of x in equation (i), $y = 6 - 2x = 6 - 2 \times (-2) = 6 + 4 = 10$

$\therefore$ When $x = -2, y = 10$ the two matrices are equal.

Exercise 1.5 (B)

1. a. If matrix A = $[a_{ij}]_{m\times n}$ then under which of the following conditions is A a square matrix?

   i. $m < n$ ii. $m > n$

   iii. $m = n$ iv. None of the above.

   b. Define the following types of matrices with an example:

   i. Square matrix ii. Diagonal matrix

   iii. Scalar matrix iv. Symmetric matrix

   c. Compare between identity matrix and scalar matrix.

2. Write the types of the given matrices:

   a. $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 2 & 6 \end{bmatrix}$

   d. $\begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}$ e. $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ f. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

3. Find the values of x, y, and z from the following conditions.

a. $\begin{bmatrix} x-1 & 2 \\ 2y & z \end{bmatrix}$ and $\begin{bmatrix} 7 & 2 \\ 10 & 5 \end{bmatrix}$ are equal matrices.

b. $\begin{bmatrix} 5x-1 & 0 \\ 2y & 3z+5 \end{bmatrix}$ is an identity matrix.

c. $\begin{bmatrix} 4 & x^2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 2z-3 & 8 \end{bmatrix}$

d. $\begin{bmatrix} 3x+1 & 0 \\ 0 & \frac{z}{3} \end{bmatrix}$ Is a scalar matrix with the element 7 on main diagonal.

e. $\begin{bmatrix} 5 & y+5 \\ 2y & 4 \end{bmatrix}$ is symmetric matrix.

f. $\begin{bmatrix} x+y & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & x-y \end{bmatrix}$

g. $\begin{bmatrix} 2x+y & x-3y \\ 2 & 2z-6 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 2 & 5z+3 \end{bmatrix}$

h. $\begin{bmatrix} Sinx & Cosy \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 3 & tanz \end{bmatrix}$, where $0 \leq x, y, z \leq 90°$

4. If $\begin{bmatrix} a-b & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & ab \end{bmatrix}$ then find the possible values of a and b.

5. If matrix A has order $(2x-3)$ and $(y+5)$ and matrix B has order $(x+2)$ and $(3y-1)$. If A and B are equal matrices, then find the order of matrix A.

Answer

1. Show to the teacher. 2. a. Scalar matrix b. Identity matrix c. Lower triangular matrix d. Row matrix e. Symmetric matrix f. Diagonal matrix

3. a. $x = 8, y = 5, z = 5$ b. $x = \frac{2}{5}, y = 0, z = \frac{-4}{3}$ c. $x = \pm 4, y = \pm 2, z = 4$
d. $x = 2, z = 21$ e. $y = 5$ f. $x = 4, y = 0$ g. $x = 6, y = -2, z = -3$
(h) $x = 30°, y = 0°, z = 45°$

4. $a = 4, b = 2$ or $a = -2, b = -4$ 5. 7×8

1.5.2 Operations on Matrix

Addition of Matrices

Activity 1

The amounts (kg) of potato, tomato, and cauliflower sales on sunday to tuesday in two shops are given in the table below:

First Shop

Day/Item	Potato	Tomato	Cauliflower
Sunday	3	4	6
Monday	5	7	10
Tuesday	8	5	11

Second Shop

Day/Item	Potato	Tomato	Cauliflower
Sunday	5	7	3
Monday	4	6	5
Tuesday	7	3	10

Study the two tables above and write answers to the following questions:

a. How many kilograms of potatoes were sold in total in the two shops on Sunday?

b. How many kilograms of cauliflower were sold in total in the two shops on Tuesday?

c. How many kilograms of each item were sold in both shops from Sunday to Tuesday?

Based on the discussion, the supply of each item in the first shop and the second shop can be written in matrix form as follows:

$$A = \begin{bmatrix} 5 & 7 & 3 \\ 4 & 6 & 5 \\ 7 & 3 & 10 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 6 \\ 5 & 7 & 10 \\ 8 & 5 & 11 \end{bmatrix}$$

Now, adding matrix A and B according to the form of matrix addition

$$A + B = \begin{bmatrix} 5 & 7 & 3 \\ 4 & 6 & 5 \\ 7 & 3 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 6 \\ 5 & 7 & 10 \\ 8 & 5 & 11 \end{bmatrix} = \begin{bmatrix} 5+3 & 7+4 & 3+6 \\ 4+5 & 6+7 & 5+10 \\ 7+8 & 3+5 & 10+11 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 9 \\ 9 & 13 & 15 \\ 15 & 8 & 21 \end{bmatrix}$$

From Sunday to Tuesday, Potato: 8 kg, 9 kg, 15 kg; Tomato: 11 kg, 13 kg, 8 kg; and Cauliflower: 9 kg, 15 kg, 21 kg were supplied respectively.

Only matrices of the same order can be added and subtracted. Similarly, in matrix addition, corresponding elements, i.e., elements in the same position, must be added. The operation of subtraction in matrices also follows a similar process as addition.

Example 1

If $A = \begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 1 \\ 7 & 3 & 0 \end{bmatrix}$, then find A + B and A – B.

Solution

Here, $A + B = \begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 1 \\ 7 & 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+2 & -5+6 & 2+1 \\ 1+7 & 5+3 & 3+0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 3 \\ 8 & 8 & 3 \end{bmatrix}$

$A - B = \begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 6 & 1 \\ 7 & 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0-2 & -5-6 & 2-1 \\ 1-7 & 5-3 & 3-0 \end{bmatrix}$

$= \begin{bmatrix} -2 & -11 & 1 \\ -6 & 2 & 3 \end{bmatrix}$

1.5.3 Properties of Matrix Addition

a. Closure Property

Consider the following example:

If $P = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, what kind of matrix is P + Q? Are the orders of P, Q, and P + Q the same?

Here, $P = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $P + Q = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3+1 & 5+3 \\ 2+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 4 & 9 \end{bmatrix}$

Since the order of matrices P and Q is 2 × 2, their sum P + Q is also a matrix whose order is also 2 × 2.

The sum of two matrices of the same order is also a matrix of the same order. This property of matrix addition is called the closure property.

Thought Provoking Question: Does the closure property hold for matrix subtraction as well? Investigate with an example and write the conclusion.

b. Commutative Property

Let us take any two matrices, for example: $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Calculating A + B and B + A, do we get the same result?

$$A + B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+2 & 1+0 \\ 2+1 & 3+4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 7 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2+1 & 0+1 \\ 1+2 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 7 \end{bmatrix}$$

Thus, A + B = B + A. Therefore, for any matrices, A + B = B + A holds. This property of matrix addition is called the commutative property.

For any matrices A and B of the same order, A + B = B + A holds. This property of matrix addition is called the commutative property.

Thought Provoking Question: Does the commutative property hold for matrix subtraction as well? Investigate with an example and write the conclusion.

c. Associative Property

Suppose, matrices $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Finding, (A + B) + C and A + (B + C)

$$(A + B) + C = \left(\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}\right) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & 5 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\left(\begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & 5 \end{bmatrix}$$

$\therefore$ (A + B)+ C = A + (B + C)

Thus, the associative property holds for matrix addition.

For any three matrices A, B, and C of the same order, (A + B) + C = A + (B + C) holds, i.e., the sum obtained by first adding A and B and then adding C is equal to the sum obtained by adding A to the sum of B and C. This property of matrix addition is called the associative property.

Thought Provoking Question: Does the associative property hold for matrix subtraction as well? Investigate with an example and write the conclusion.

d. Existence of Additive Identity

Suppose, $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now,

$$A + O = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1+0 \\ 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = A$$

$$O + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0+1 & 0+1 \\ 0+2 & 0+3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = A$$

Thus, for the above matrix A, there exists a zero matrix of the same order such that A + O = O + A = A. This property of matrix addition is called the existence of additive identity.

For any matrix A, there exists a zero matrix O of the same order such that A + O = O + A = A. This property of matrix addition is called the existence of additive identity.

e. Existence of Additive Inverse

Take a matrix, $A=\begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix}$, can we find another matrix of the same order such that when added to A gives the zero matrix?

Yes, it is possible. For example, another matrix $B = \begin{bmatrix} 0 & 5 & -2 \\ -1 & -5 & -3 \end{bmatrix}$

$$\text{Now, } A + B = \begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -2 \\ -1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$B + A = \begin{bmatrix} 0 & 5 & -2 \\ -1 & -5 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -5 & 2 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, A + B = B + A = O (Zero matrix)

Since adding the two matrices above gives the zero matrix, these two matrices are called additive inverses of each other.

For any matrix A, there exists another matrix B of the same order such that A + B = B + A = O (Zero matrix), then B is called (–A). This property of matrix addition is called the existence of additive inverse.

In such a case, A and B are called additive inverses of each other. That is, for any matrix A, its additive inverse is another matrix of the same order (–A).

1.5.4 Scalar Multiplication of Matrix

$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. What can you say about 2A?

$$2A = A + A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 2A$$

When a matrix A is multiplied by any scalar k, all elements of A must be multiplied by k.

For example: If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Example 1

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$, find the value of 3A.

Solution

Here, the given matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

Now, $3A = 3\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 12 & 6 \end{bmatrix}$

$\therefore 3A = \begin{bmatrix} 6 & 9 \\ 12 & 6 \end{bmatrix}$

Example 2

If $A = \begin{bmatrix} 1 & 3 \\ -9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} x & 2y \\ 9 & 0 \end{bmatrix}$ and A + B is a zero matrix of the same order, then find the values of x and y.

Solution

Here A + B is a zero matrix, so

$$\begin{bmatrix} 1 & 3 \\ -9 & 0 \end{bmatrix} + \begin{bmatrix} x & 2y \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Or, $\begin{bmatrix} 1+x & 3+2y \\ -9+9 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Or, $\begin{bmatrix} 1+x & 3+2y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, equating corresponding values,

$1 + x = 0$, $x = -1$, Also, $3 + 2y = 0$ Or, $2y = -3$ Or, $y = \frac{-3}{2}$

Exercise 1.5 (C)

1. What are the necessary conditions for matrix addition and subtraction? Write them.
2. a. What are the properties of matrix addition? Write them.
   b. If A and B are matrices of the same order, which property of matrix addition does A + B = B + A represent ?
   c. Write the associative property of matrix addition.
3. Which of the following matrices can be added? State with reason:

   a. $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ b. $P = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -4 & 2 \end{bmatrix}$ c. $Q = [1 \quad 2 \quad -2]$

   d. $R = \begin{bmatrix} 3 & 7 \\ 2 & 5 \\ -4 & 2 \end{bmatrix}$ e. $X = \begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}$

4. If $X = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}$, then find the following matrices:

a. $X + Y$ b. $X - Y$ c. $2X + 3Y$ d. $\frac{1}{2}X - 2Y$

5 a. If $X = \begin{bmatrix} -2 & 2 \\ 8 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} -4 & -5 \\ 7 & 0 \end{bmatrix}$, then verify the commutative property of matrix addition. Does the commutative property hold for matrix subtraction as well? Write and verify your conclusion.

b. If $P = \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix}$, $Q = \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 4 & 8 \\ 3 & 0 \end{bmatrix}$, then verify the associative property of matrix addition and prove it.

c. If $M = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 5 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 5 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 5 \end{bmatrix}$, then prove the

b. following:

(i) $M + H = H + M$ (ii) $(M + H) + N = M + (H + N)$

(iii) $M + (-M) = O$ (O is a zero matrix of the same order.)

6. Find the values of x and y in each of the following cases.

a. $\begin{bmatrix} 1 & 4x \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 4 & 3y \end{bmatrix}$

b. $\begin{bmatrix} 1 & 3x \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & y \end{bmatrix} =$ I is an identity matrix of the same order.

c. The additive inverse of $\begin{bmatrix} x+5 & 2 \\ 0 & 2y-3 \end{bmatrix}$ is $\begin{bmatrix} 4 & 2 \\ 0 & -7 \end{bmatrix}$

d. $2\begin{bmatrix} 5 & 4x \\ 7 & y-3 \end{bmatrix} + 3\begin{bmatrix} 4 & 1 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 4 & 3y \end{bmatrix}$

e. $2\begin{bmatrix} 3x & 0 \\ 0 & y-3 \end{bmatrix} + 3\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ is a zero matrix of the same order.

7.

a. If $P = \begin{bmatrix} 3 & 0 \\ 4 & -2 \end{bmatrix}$ and $3P + X = \begin{bmatrix} 8 & 6 \\ 9 & 6 \end{bmatrix}$, then find the matrix X.

b. If $P = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$ and $R = \begin{bmatrix} 4 & 8 \\ 3 & 0 \end{bmatrix}$, then find the matrix X such that

$2P - 3X = 3R$

8. If matrix $P + Q = \begin{bmatrix} 5 & 0 \\ 3 & -2 \end{bmatrix}$ and $P - Q = \begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix}$, then find the matrices P and Q.

9. If matrix $X = \begin{bmatrix} -3 & -1 \\ 6 & 7 \end{bmatrix}$, then find the additive inverse of matrix X.

Answer

1. Same order
2. a. Show to the teacher. b. Commutative property

   c. If A, B, and C are matrices of the same order, then (A + B) + C = A + (B + C) is called the associative property.
3. Matrices A and X (same order), Matrices P and R (same order)
4. a. $X + Y = \begin{bmatrix} 5 & 11 \\ 15 & 0 \end{bmatrix}$ b. $X - Y = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ c. $2X + 3Y = \begin{bmatrix} 11 & 27 \\ 37 & 0 \end{bmatrix}$ d. $\frac{1}{2}X - 2Y = \begin{bmatrix} 0 & -7 \\ -10 & 0 \end{bmatrix}$
5. a. The commutative property does not hold for matrix subtraction. b. and c. Show to the teacher.
6. a. $x = \frac{3}{2}, y = 3$ b. $x = \frac{-2}{3}, y = -2,$ c. $x = -9, y = 5$ d. $x = \frac{5}{8}, y = 12$ e. $x = -2, y = -6$
7. a. $X = \begin{bmatrix} -1 & 6 \\ -3 & 12 \end{bmatrix}$ b. $X = \begin{bmatrix} \frac{-10}{3} & -8 \\ -3 & \frac{-4}{3} \end{bmatrix}$ 8. $P = \begin{bmatrix} \frac{13}{2} & 2 \\ \frac{5}{2} & 2 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{-3}{2} & -2 \\ \frac{1}{2} & -4 \end{bmatrix}$
9. $\begin{bmatrix} 3 & 1 \\ -6 & -7 \end{bmatrix}$ 10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Identity matrix

Project Work

1. Form a group of five friends and go to two nearby stationery shops. Write the stock count of any five similar types of stationery items in the form of a matrix. Write down the matrix operations used to find the total stock count of those items in both stationery shops and the number of items remaining after sales on that same day. Using those operations, find the remaining stock and prepare a report to present in the classroom.
2. Form a group of five friends and go to the school canteen. Collect data on the different quantities (in kg) of food items wasted (like food, vegetables, and others) over any three consecutive days. Write this data in matrix form. Using matrix operations, find out the total quantity of food wasted each day. Prepare a report in the specified format and present it in the classroom.
3. Each student should open the internet browser (such as Google Chrome, Firefox, Internet Explorer, etc.) available on their mobile or computer at home, search for any five uses of matrices, and present them in the classroom.

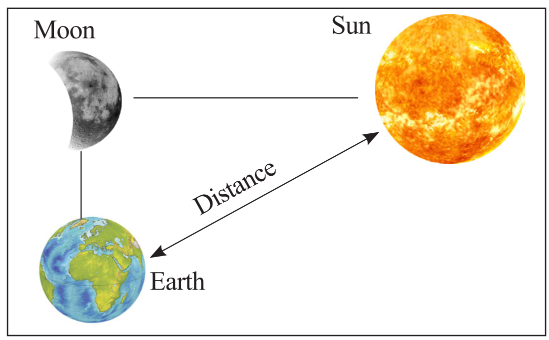
Lesson 2

Trigonometry

Trigonometry is a branch of mathematics that deals with the relationships between three sides and angles of a triangle. Hipparchus is considered as the father of trigonometry because he was the one who compiled the first trigonometric table.

Although the use of trigonometry began from astronomy and geography, it is prevalent in fields like physics, navigation, and engineering at present. For example, trigonometry is used in tasks such as calculating the height of building and mountains, surveying bridge construction, and in construction of roads etc.

Can we find the height of the mountain without climbing it?



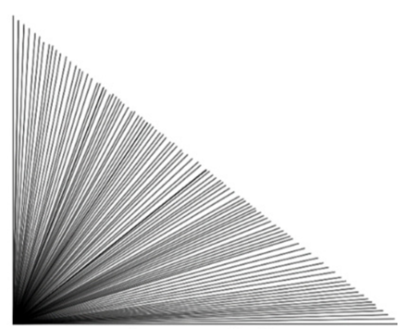
How can we find the distance from earth to sun and earth to moon?

2.1. Measurement of Angles

a. Sexagesimal System

In this system, angles are measured in degrees (°). When a straight line rotates from its initial position and makes one complete rotation, it forms an angle of 360°. A right angle is 90°.

Each 1° is divided into 60 minutes (') and each 1 minute is divided into 60 seconds (").



1 right angle = 90° (degree)

Relations:

1 right angle = 90° (Degrees)

1° (Degree) = 60' (Minutes)

1' (Minute) = 60" (Seconds)

b. Centesimal System

In this system, angle is measured in grade (g). When a straight line rotates from its initial position and makes one complete rotation, it forms an angle of 400^g and a right angle is equal to 100 grades. Each grade is divided into 100 equal parts, called "minutes" ('), and each minute is divided into 100 equal parts, called seconds (").

1 right angle = 90° (degree)

Relations:

1 right angle = 100^g (grades)

1g (Grade) = 100' (minutes)

1' (Minute) = 100" (seconds)

Example 1

a. Convert 60°15'30" to sexagesimal seconds.

b. Convert 20°10'12" to degrees.

Solution

a. Here, 60°15 30" = (60 × 60 × 60 + 15 × 60 + 30)"

= (216000 + 900 + 30)" = 216930"

b. 20°10'12"

$$= \left(20 + \frac{10}{60} + \frac{12}{60 \times 60}\right)^\circ = \left(20 + \frac{1}{6} + \frac{1}{300}\right)^\circ$$

$$= \left(\frac{20 \times 300 + 50 + 1}{300}\right)^\circ = \left(\frac{6000 + 50 + 1}{300}\right)^\circ = \left(\frac{6051}{300}\right)^\circ = 20.17^\circ$$

Example 2

a. Convert 27^g 50' 60" to centesimal seconds.

b. Convert 45^g 40' 90" to grades.

Solution

a. 27^g 50' 60" = (27 × 100 × 100 + 50 × 100 + 60)" = (270000 + 5000 + 60)" = 275060"

b. 45^g 40' 90"

$$= \left(45 + \frac{40}{100} + \frac{90}{100 \times 100}\right)^g$$

$$= \left(45 + \frac{4}{10} + \frac{9}{1000}\right)^g = \left(\frac{45000 + 400 + 9}{1000}\right)^g = \left(\frac{45409}{1000}\right)^g = 45.409^g$$

2.1.2 Conversion between Sexagesimal and Centesimal System

The conversion between the sexagesimal and centesimal systems can be done as follows:

Method of Conversion from Sexagesimal ato Centesimal	Method of Conversion from Centesimal to Sexagesimal
We know, 1 right angle = 90° (in degrees) 1 right angle = 100^g (in grades) Therefore, 90° = 100^g $1° = \left(\frac{100}{90}\right)^g = \left(\frac{10}{9}\right)^g$ Therefore, if an angle's degree value is D°, then $D° = \left(\frac{10}{9} \times D\right)^g$	We know, 1 right angle = 90° (in degrees) 1 right angle = 100^g (in grades) Therefore, 100^g = 90° $1^g = \left(\frac{90}{100}\right)^\circ = \left(\frac{9}{10}\right)^\circ$ Therefore, if an angle's grade value is Gg, then $G^g = \left(\frac{10}{9} \times G\right)^\circ$

Example 3

Convert the angle 81° to the centesimal system (grades).

Solution

Here, given angle = 81°

We know that $1° = \left(\frac{10}{9}\right)^g$

Now, $81° = \left(\frac{10}{9} \times 81\right)^g = (10 \times 9)^g = 90^g$

Hence, $81° = 90^g$

Example 4

Convert the angle 60^g to the sexagesimal system (degrees).

Solution

Here, given angle $= 60^g$

We know that, $1^g = \left(\frac{9}{10}\right)^\circ$

Now, $60^g = \left(\frac{9}{10} \times 60\right)^\circ = (9 \times 6)^\circ = 54^\circ$

Hence, $60^g = 54^\circ$

Example 5

Convert 64°51'45" to the centesimal system (grade).

Solution

Here, = 64°51'45"

Now, $64°51'45" = 64° + \left(\frac{51}{60}\right)^{\circ} + \left(\frac{45}{60 \times 60}\right)^{\circ}$

$= 64° + 0.85° + 0.0125°$

$= 64.8625°$

Now, for converting into grade, we know $1° = \left(\frac{10}{9}\right)^{g}$

Therefore, $64.8625° = \left(\frac{10}{9} \times 64.8625°\right)^{g}$

$= 72.06944^{g}$

$= 72^{g}\ (0.06944 \times 100)' = 72^{g}\ 6.944'$

$= 72^{g}\ 6'\ (0.944 \times 100)"$

$= 72^{g}\ 6'\ 94.4"$

Hence, $64°51'45" = 72^{g}\ 6'\ 94"$

Example 6

Convert 56^{g} 87' 50" to the sexagesimal system.

Solution

Here, = 56^{g} 87' 50"

Now, $56^{g}\ 87'\ 50" = 56^{g} + \left(\frac{87}{100}\right)^{g} + \left(\frac{50}{100 \times 100}\right)^{g}$

$= 56^{g} + 0.87^{g} + 0.005^{g}$

$= 56.875^{g}$

Now, we know $1^{g} = \left(\frac{9}{10}\right)^{\circ}$

Therefore, $56.875^{g} = \left(\frac{9}{10} \times 56.875\right)^{\circ}$

$= 51.1875°$

$= 51°(0.1875 \times 60)'$

$= 51°\ 11'\ (0.25 \times 60)"$

$= 51°\ 11'\ 15"$

Thus, 56^{g} 87' 50" = 51° 11' 15"

Example 7

The sum of two angles is 100° and their difference is 20^g. Find the angles in degrees.

Solution

Let the two angles be x and y.

We know, $1^g = \left(\frac{9}{10}\right)^\circ$

$20^g = \left(\frac{9}{10} \times 20\right)^\circ = 18^\circ$

According to first condition, $x + y = 100^\circ$

Or, $x = 100^\circ - y$ ……………………(i)

From the second condition,

$x - y = 18^\circ$ $\{\because 20^g = \left(\frac{9}{10} \times 20\right)^\circ = 18^\circ\}$

Or, $x = 18^\circ + y$ ……………………(ii)

Substituting the value of x form equation (i) in equation (ii)

Or, $100^\circ - y = 18^\circ + y$

Or, $100^\circ - 18^\circ = y + y$

Or, $2y = 82^\circ$

Or, $y = \frac{82^\circ}{2}$

Or, $y = 41^\circ$

Again, substituting the value of y in the equation (i)

Or, $x = 100^\circ - 41^\circ$

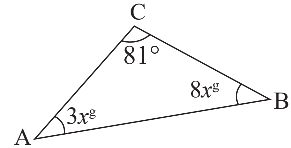
Or, $x = 59^\circ$

Hence, required angles are 59° and 41°.

Example 8

Two angles of a triangle are in the ratio 3:8 and third angle 81°. Convert all angles of the triangle in grade.

Solution



In the triangle ABC,

Let first angle (∠CAB) = $(3x)^g$

Second angle (∠ABC) = $(8x)^g$

Third angle (∠ACB) = $81^\circ = \left(\frac{10}{9} \times 81\right)^g = 90^g$

We know, in ΔABC

$3x + 8x + 90^g = 200^g$

Or, $11x = 200^g - 90^g$

Or, $11x = 110^g$

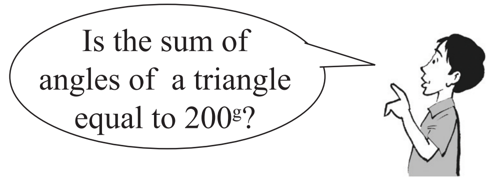
Or, $x = \frac{110^g}{11} = 10^g$

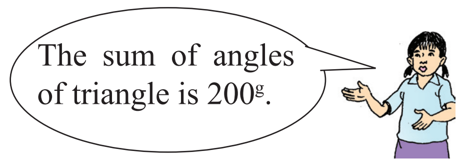
Therefore,

First angle $= 3 \times 10^g = 30^g$

Second angle $= 8 \times 10^g = 80^g$

Third angle $= 90^g$





Exercise 2.1 (A)

1. Fill in the blanks:

   a. 1 right angle = ….. (Degrees) b. 1 right angle = ….. (Grades)

   c. 2 right angles = …… (Grade) d. 200^g = …… (Degree)

   e. $1^\circ = \left(\frac{\dots}{\dots}\right)^g$ (f) $1^g = \left(\frac{\dots}{\dots}\right)^\circ$

2. Convert into seconds (sexagesimal):

   a. 35' b. 50' 40" c. 30° 40' 50"

   d. 55° 30' 10" e. 10° 25' 48" f. 55° 56' 28"

3. Define sexagesimal and centesimal systems and where are they used? How is conversion done? Explain with examples.

4. Convert into sexagesimal (degrees):

   a. 30^g 20' 10" b. 25^g 15' 10" c. 45^g 35' 25"

   d. 30^g 12' e. 26^g 15" f. 47^g 48' 49"

5. Convert into centesimal (grades):

   a. 25° 45' 30" b. 30° 15' 15" c. 49° 50' 25"

   d. 44° 35' 25" e. 80° 50' 20" f. 76° 26' 33"

6. Convert into degrees:

   a. 50^g b. 80^g c. 130^g d. 160^g e. 70^g f. 250^g

7. Convert into grades:

   a. 50^g 40' 8" b. 40^g 32' 33" c. 56^g 85' 50"

   d. 45^g 35" e. 37^g 50' f. 98^g 42' 37"

8. Convert into grades:

a. 45° b. 270° c. 18° d. 36° e. 108° f. 54°

9. In a right-angled triangle, one angle is 60°.

a. What is the measure of the angle (in degrees) that makes a triangle a right-angled triangle?

b. What must be the largest angle in a right-angled triangle in grades?

c. Find the measure of the remaining angle in degrees.

10. The three angles of a triangle are in the ratio 1:2:3. Find each angle in degrees.

11. The three angles of a triangle are in the ratio 2:3:4.

a. Find their values in degrees.

b. Find their values in grades.

12. The angles of a triangle are in the ratio 5:7:8.

a. Find their values in degrees.

b. Find their values in grades.

13. The four angles of a quadrilateral are in the ratio 3:4:5:6.

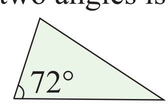
a. Find their measurement in degrees.

b. Convert the smallest and largest angles into grades and find their difference.

14. In a triangle, one angle is 72°. The ratio of the remaining two angles is 1:3.

a. Find the value of remaining two angles in degrees.

b. Find all the angles of the triangle in grades.



c. What should the degree measures of the remaining two angles for their ratio to be 1:5?

15. Two angles of a triangle are in the ratio 3:4 and the third angle is 60^g.

a. How many degrees are there in 60^g?

b. Find all three angles of the triangle in degrees.

c. Based on the angles, what type of triangle is it? Write with reasons.

16. One acute angle of a right-angled triangle is $\frac{3}{10}$ of a right angle.

a. Write in number of degree for a right angle.

b. How many degrees are $\frac{3}{10}$ of a right angle?

c. Find the value of remaining acute angle in grade.

17. The sum of two angles is 100° and their difference is 20^g.
    a. How many grades are equal to 1°?
    b. Find the values of the two angles in degrees.
    c. Find the values of the two angles in grades.
18. The sum of two angles is 45° and their difference is 30^g.
    a. How many degrees are equal to 1g?
    b. Find the values of the two angles in degrees.
    c. Find the values of the two angles in grades.

Answer

1. a. 90° b. 100^g c. 200^g d. 180° e. $\left(\frac{10}{9}\right)^g$ (f) $\left(\frac{9}{10}\right)^\circ$

2. a. 2100" b. 3040" c. 110450" d. 19810" e. 37548" f. 201388"

3. Show to the teacher. 4. a. 27°10'51.24" b. 22°38'9.24" c. 40°49'21"

d. 27°6'28.8" e. 23°24'48.6" f. 42°44'11.07" 5. a. 28^g62'37.03" b. 33^g61'57.40"

c. 55^g37'80.86" d. 49^g54'47.53" e. 89^g82'98.76" (f) 84^g93'61.11" 6. a. 45°

b. 72° c. 117° d. 144° e. 63° f. 225° 7.a. 50.4^g b. 40.32^g

c. 56.85^g d. 45.0035^g e. 37.5^g (f) 98.42^g 8.a. 50^g b. 300^g

c. 20^g d. 40^g e. 120^g f. 60^g 9. a. 90° b. 100^g

c. 30° 10. 30°, 60°,90° 11. a. 40°, 60°, 80° b. 44.44^g, 66.66^g, 88.88^g

12. a. 45°, 63°, 72° b. 50^g, 70^g, 80^g 13.a. 60°, 80°, 100°, 120° b. $\left(\frac{600}{9}\right)^g$

14. a. 27°, 81° b. 30^g, 80^g, 90^g c. 18°, 90° 15. a. 54° b. 54°, 54°, 72°

c. Isosceles triangle 16. a. 90° b. 27° c. 70^g 17. a. $\left(\frac{10}{9}\right)^g$

b. 41°, 59° c. $\left(\frac{590}{9}\right)^g$, $\left(\frac{410}{9}\right)^g$ 18. a. $\left(\frac{9}{10}\right)^\circ$ b. 9°, 36° c. 10^g, 40^g

2.1.3 System of Circular Measure

Activity 1

Sit in an appropriate group. Draw a circle with center O by taking radius greater than 3 cm. From the center O, draw a radius to a point A on the circumference. Take a rope to measure an arc equal in length to the radius from point A on the circumference, mark another point B where this arc meets the circumference. Join point B with the center O.

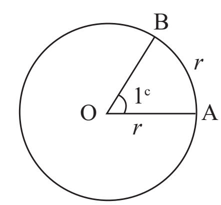
Now, measure is the angle AOB formed at the center of the circle. This angle is called (1^c) radian.

The circular measure is the standard method of measuring angles. In this system, angles are measured in the unit radian c.

An angle of 1 radian (1^c) is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle.

Thus, the radian is the standard unit of angular measurement. In the given circle with center O and radius OA = r, the arc AB is equal in length to the radius OA.

Thus, $OA = \widehat{AB} = r$ Thus, $\angle AOB = 1^c$



When a rotating line makes one complete rotation around the center, the angle formed is $2\pi^c$. Thus, the complete angle formed at the center of a circle is $2\pi^c$.

2.1.4 Theorem on System of Radian Measure

In the circular measurement system, there are some principles that are universally accepted. These principles have been established and verified as theorems.

Theorem 1: Radian is a constant angle.

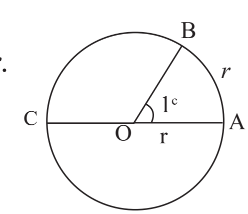
Proof: There is a circle with center O and radius OA = r. The arc AB is equal in length to the radius OA. Therefore, $OA = \widehat{AB} = r$. By taking an arc AB whose length is equal to the radius r, draw the central angle AOB. According to the definition of a radian, $\angle AOB = 1^c$.

Extend the radius AO to a point C to form the diameter AC.

According to the formula, the circumference of the circle $c. = 2\pi r$.

Now, circumference of semi-circle $(\widehat{ABC}) = \frac{1}{2} \times 2\pi r = \pi r$

Central angle $\angle AOC$ = straight angle = $180°$



Now, the ratio of central angles is equal to the ratio of their corresponding arcs.

$\frac{\angle AOB}{\angle AOC} = \frac{\widehat{AB}}{\widehat{ABC}}$ [∵ From the relationship between the central angle and its corresponding arc]

or, $\frac{1^c}{180} = \frac{r}{\pi r}$ [∵ $\angle AOC = 180°$ and $\widehat{ABC} = \pi r$, circumference of semi-circle]

or, $1^c = \frac{180°}{\pi}$

The value of 1^c is independent of the radius (r). Since both 180° and π are both constant, Therefore, 1^c is also a constant angle.

Activity 2

Study the table below to understand the relationship between degree, grade, and radian and fill in the blanks:

Angle	Degree	Grade	Radian
One complete turn	....	400^g	$2\pi^c$
One right angle	$\frac{360°}{4} = 90°$	....	$\frac{2\pi^c}{4} = \frac{\pi^c}{2}$
Two right angle	180°	....	....

2.1.5 The Relationship among Length of Arc, Radius and Central Angle

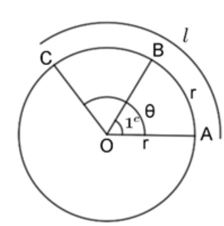
Activity 3

Draw a circle with center O by taking a radius greater than 4 cm. Take two arcs AB and CD on the circle. Construct the central angles AOB and COD based on the arcs AB and CD respectively. Using a protractor, measure the angles AOB and COD and find their ratio.

Similarly, with the help of a thread and a ruler, measure the lengths of arcs AB and CD and find their ratio.

Is the central angle formed based on the fact that it is equal to its corresponding arc in degree measure? Discuss.

Theorem 2: In a circle with radius r units, an arc of length *l* units subtends a central angle θ given by $\theta = \left(\frac{l}{r}\right)^c$.



Proof: Suppose ABC is a circle with center O and radius OA = r. Two central angles ∠AOB and ∠AOC are subtended on the different arcs AB and ABC respectively. $\widehat{AB} = OA = r$. According to the definition of a radian, $\angle AOB = 1^c$.

Here, $\widehat{ABC} = l$ and $\angle AOC = \theta$

Now, the ratio of central angles is equal to the ratio of their corresponding arcs.

Therefore, $\frac{\angle AOC}{\angle AOB} = \frac{\overset{\frown}{ABC}}{\overset{\frown}{AB}}$

Or, $\frac{\theta}{1^c} = \frac{l}{r}$

$\therefore \theta = \left(\frac{l}{r}\right)^c$

The central angle $(\theta) = \left(\frac{\text{The arc subtended by the angle at the center of the circle}}{\text{Radius of circle}}\right)$ radian

Example 1

Convert the given angle 75° into the radian system.

Solution

Here, given angle = 75°

We know that, $1^\circ = \left(\frac{\pi}{180}\right)^c$

Thus, $75^\circ = \left(\frac{\pi}{180} \times 75\right)^c = \left(\frac{5\pi}{12}\right)^c$

Thus, $75^\circ = \left(\frac{5\pi}{12}\right)^c$

Example 2

Convert the given angle $\frac{\pi^c}{4}$ into degree.

Solution

Here, given angle = $\frac{\pi^c}{4}$

We know that, $1^c = \left(\frac{180}{\pi}\right)^\circ$

Now, $\frac{\pi^c}{4} = \left(\frac{180}{\pi} \times \frac{\pi}{4}\right)^\circ = 45^\circ$

Thus, $\frac{\pi^c}{4} = 45^\circ$

Example 3

Convert the given angle 90^g into the radian system.

Solution

Here, given angle $= 90^g$

We know that, $200^g = \pi^c$

$$1^g = \frac{\pi^c}{200}$$

Now, $90^g = \left(\frac{\pi}{200} \times 90\right)^c$

Thus, $90^g = \frac{9\pi^c}{20}$

Example 4

Convert the given $\frac{5\pi^c}{8}$ into grade.

Solution

Here, given angle $= \frac{5\pi^c}{8}$

We know that, $\pi^c = 200^g$

$$1^c = \frac{200^g}{\pi}$$

Now, $\frac{5\pi^c}{8} = \left(\frac{200}{\pi} \times \frac{5\pi}{8}\right)^g = 125^g$

Thus, $\frac{5\pi^c}{8} = 125^g$

Example 5

In a right-angled triangle, the difference between the two acute angles is $\frac{\pi^c}{9}$.

a. What is the measure of angle $\frac{\pi^c}{9}$ in degrees?

b. Taking the two acute angles as $x°$ and $y°$, write the two equations thus formed.

c. Find the measures of the two acute angles in degrees.

Solution

a. Converting $\frac{\pi^c}{9}$ in degree $\frac{\pi^c}{9} = \left(\frac{\pi}{9} \times \frac{180}{\pi}\right)^\circ = 20°$

b. According to question, $x + y + 90° = 180°$

Or, $x + y = 90°$ .... (i)

∵

and $x - y = \frac{\pi^c}{9}$

Or, $x - y = 20°$ .... (ii) $\{\because \frac{\pi^c}{9} = 20°\}$

c. Now, adding equation (i) and (ii)

$$\begin{array}{l} x + y = 90° \\ x - y = 20° \\ \hline 2x = 110° \end{array}$$

Or, $x = \frac{110°}{2}$

Or, $x = 55°$

Again, substituting the value of x in equation (i)

$55° + y = 90°$

Or, $y = 90° - 55°$

Or, $y = 35°$

Thus, the required angles are 55° and 35°.

Example 6

What is the angle (in radians) between the minute hand and the hour hand of a clock when the time is 2:30? Find it.

Solution

At 2:30, the minute hand of the clock is exactly at 6, and the hour hand is between 2 and 3. Thus, 2:30 = 2 hours + 30 minutes = (2 hours + $\frac{30}{60}$ hours) = 2 hours + 0.5 hours = 2.5 hours.

Now, we know that:

The hour hand of the clock makes an angle in 12 hours = $2\pi^c$

The hour hand of the clock makes an angle in 1 hours = $\frac{2\pi^c}{12}$

Therefore, the hour hand of the clock make an angle in 2.5 hours = $\frac{2\pi^c}{12} \times 2.5 = \frac{5\pi^c}{12}$

Similarly, the minute hand of the clock has already passed 30 minutes.

The minute hand of the clock makes an angle in 60 minute = $2\pi^c$.

The hour hand of the clock makes an angle in 1 minute = $\frac{2\pi^c}{60}$.

The hour hand of the clock makes an angle in 30 minute = $\frac{2\pi^c}{12} \times 30 = \pi^c$.

Hence, the angle between the hour hand and the minute hand = $\pi^c - \frac{5\pi^c}{12} = \frac{12\pi^c - 5\pi^c}{12} = \frac{7\pi^c}{12}$

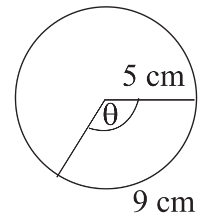
Note: Can this problem be solved in a different way? If so, solve it and compare.

Example 7

In the given circle, radius 5 cm and arc of 9 cm make an angle θ at the centre.

a. Write the formula to find the central angle (θ).

b. Find the central angle (θ) in radian system.



Solution

Here, length of radius (r) = 5 cm

Length of arc (l) = 9 cm

Central angle (θ) = ?

a. Formula to find the central angle (θ) = $\left(\frac{l}{r}\right)^c$

b. Now, according to formula, central angle (θ) = $\left(\frac{l}{r}\right)^c = \left(\frac{9 \text{ cm}}{5 \text{ cm}}\right)^c = 1.8^c$

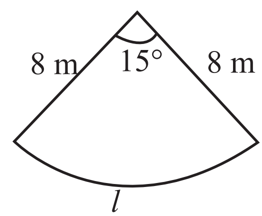
Example 8

A girl is playing on a swing, and an angular displacement of 15° is formed. The length of the swing is given as 8 m.

a. Convert 15° to radians.

b. What distance does the girl cover in one swing from the center to one side? Find it.

c. Why you use radians measure while calculating distance in part (b).



Solution

Here, length of swing = radius of circle (r) = 8 m

a. Convert the central angle (θ) = 15° into radian, $15° = \left(\frac{\pi}{180} \times 15\right)^c = \left(\frac{\pi}{12}\right)^c$

b. Here, the distance covered by girl in one swing from the center to one side (l) = ?

Now, central angle (θ) = $\left(\frac{l}{r}\right)^c$

Or, $\left(\frac{\pi}{12}\right)^c = \left(\frac{l}{8}\right)^c$

Or, $\frac{22}{7 \times 12} = \frac{l}{8}$

Or, $l = \frac{8 \times 22}{7 \times 12}$ Or, $l = 2.095\ m$

∴ The girl covers a distance of 2.095 m on the swing from the center to one side in one swing.

c. Central angle (θ) should always be in radian while using the formula (θ) = $\left(\frac{l}{r}\right)^c$. So, 15° should convert in radian.

Exercise 2.1 (B)

1. Fill in the blanks:

a. 2 right angle = …… radian

b. 200 grade =………….. radian

c. π^c =………………degree

d. 1 radian = …………… degree

e. 1^o =……………….radian

(f) 1^g =…………………….radian

2. What do you mean by a radian? Write.

3. What is the circular measurement system? In which unit is the angle measured in this system? Write.

4. What do l and r represent in the central angle $(\theta) = \left(\frac{l}{r}\right)^c$? Write it.

5. Study the above activities 2 and 3 and write down the conclusions you have drawn.

a. Conclusions from activity 2: ....

b. Conclusions from activity 3: ....

6. Convert the given angles into radians:

a. $30°$ b. $45°$ c. 50^g

d. 70^g e. $120°$ (f) 150^g

7. Convert the given angles into degree:

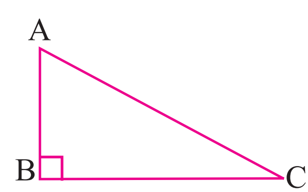
a. $\frac{\pi^c}{2}$ b. $\frac{3\pi^c}{2}$ c. $\frac{7\pi^c}{50}$ d. $\frac{3\pi^c}{4}$ e. $\frac{2\pi^c}{3}$

f. $\frac{5\pi^c}{6}$ g. $\frac{4\pi^c}{9}$ h. $\frac{5\pi^c}{12}$ i. $\frac{\pi^c}{9}$

8. Convert the given angles into grade:

a. $\frac{\pi^c}{5}$ b. $\frac{3\pi^c}{10}$ c. $\frac{4\pi^c}{25}$ d. $\frac{\pi^c}{4}$

e. $\frac{\pi^c}{8}$ f. $\frac{3}{2}\pi^c$



9. A right angled triangle ABC is given.

a. What is the degree of ∠ABC in that right angled triangle ABC? Write it.

b. In the figure, what is the value of $\frac{3}{5}$ of ∠ABC in radian?

c. In the figure, what is the value of 40% of ∠ABC in radian? Find it.

10. If one angle of a right triangle is 60° then,

a. Write the relationship between degree and radian.

b. Find the value of the remaining angles in radian.

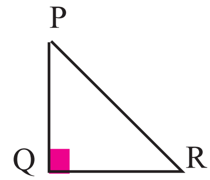
c. What is the difference between acute angles in radian? Find it.

11. The value of an acute angle in a right angled triangle is 50°.

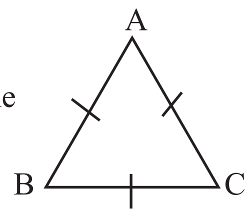
a. Find the value of the remaining angle in radians.

b. What is the difference between acute angles in radian? Find it.

c. What is the one third of the straight angles in radians?



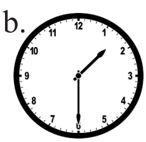
12. a. The difference between two acute angles ∠QPR and ∠PRQ of right angled triangle PQR is $\frac{3\pi^c}{10}$. Find all the angles in grade.



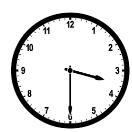
b. In the given equilateral triangle ABC, find all the angles in system of circular measure.

13. In each of the following cases, find the value of the angle between the hour hand and the minute hand of the clock in circular measure systems.

a. 
4:00

b. 
1:30

c. 
6:45

d. 
3:30

14. a. If an arc of 44 cm of a circle makes an angle of 60° at the center, find the radius of the circle.

b. If an arc of 18 cm of a circle makes an angle of 81° at the center, find the radius of that circle.

c. If an arc of 15 cm of a circle makes an angle of $\frac{3\pi^c}{4}$ at the center, find the radius of that circle.

15. a. Find the angle in degrees made at the center by an arc of 11 cm in a circle of radius 18 cm.

b. What is the measure of the angle (in degrees) subtended at the center of a circle of radius 7.254 cm by an arc 3.8 cm?

16. A clock's minute hand is 3 cm long. How much distance does the tip of the hand cover in 20 minutes?

17. A cow is tied to a peg with a rope 10 m long. When the rope is stretched and the cow moves around, the rope makes an angle of $\frac{7\pi^c}{18}$ radians from its initial position.

a. Convert $\frac{7\pi^c}{18}$ into degree.

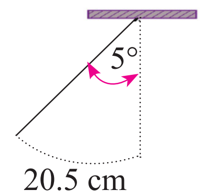
b. Find the distance covered by the cow when it makes an angle of $\frac{7\pi^c}{18}$ radians from the initial position.

18. A goat is tied to a peg with a 14 m long rope. While the rope remains stretched, it makes an angle of 70° at the peg as the goat moves around.

a. Convert 70° into radians.

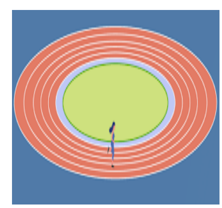
b. Find the distance covered by the goat when it makes an angle of 70°.

c. If the distance covered by the goat becomes double, find the angle (in degrees) made by the rope at the peg.



19. A pendulum swings through a distance of 20.5 cm from mean position making an angle of 5°. Find the length of pendulum.

20. A person walks along a circular path at a speed of 100 meter per minute and makes an angle of 56° at the center in 36 seconds.



a. Find the distance covered by the person in 36 seconds.

b. Find the radius of the path.

c. Find the circumference of the circular path.

21. If D, G and C respectively give the degree, grade and redian values of an angle, then prove that: $\frac{D}{180} = \frac{G}{200} = \frac{C}{\pi}$

Project work

1. State the relationship among degree, grade, and radian; and prepare a conversion chart showing how to change from one system to another. In which countries of the world is each of these systems degree, grade, or radian first introduced and used? Which system is used in mathematics in our country? Why is it necessary to know about all these systems? Does a deep understanding of degree, grade, and radian make it easier to study mathematics in any country of the world, or does it not make much difference? Why do you think grade and radian are introduced in Optional Mathematics of Grade 9? Prepare a report based on these questions, and present your findings in the classroom using chart paper, PowerPoint, or any other suitable method.

2. Find the location of your school in degrees, minutes and seconds with the help of the internet (using Google Maps or any other tool).

Answer

1 – 5. Show to the teacher.

6. a. $\frac{\pi^c}{6}$ b. $\frac{\pi^c}{4}$ c. $\frac{\pi^c}{4}$ d. $\frac{7\pi^c}{20}$ e. $\frac{2\pi^c}{3}$ f. $\frac{3\pi^c}{4}$

7. a. 90° b. 270° c. 25.2° d. 135° e. 120° f. 150° g. 80° h. 75° i. 20°

8. a. 40^g b. 60^g c. 32^g d. 50^g e. 25^g (f) 300^g

9. a. 90° b. $\frac{3\pi^c}{10}$ c. $\frac{\pi^c}{5}$ 10. a. $180° = \pi^c$ b. $\frac{\pi^c}{6}$ c. $\frac{\pi^c}{6}$

11. a. $\frac{2\pi^c}{9}$ b. $\frac{\pi^c}{18}$ (ग) $\frac{\pi^c}{3}$ 12. a. 100^g, 80^g, 20^g b. $\frac{\pi^c}{3}$

13. a. $\frac{2\pi^c}{3}$ b. $\frac{3\pi^c}{4}$ c. $\frac{3\pi^c}{8}$ d. $\frac{5\pi^c}{12}$

14. a. 42 cm b. 12.73 cm c. 6.36 cm

15. a. 35° b. 30° 16. 6.28 cm

17.a. 70° b. 12.22 m

18. a. $\frac{7\pi^c}{18}$ b. 17.11 m c. 139.99 cm

19. a. 234.82 cm 20. a. 60 m b. 61.36 m c. 385.71 m

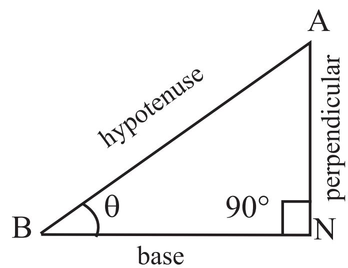
21. Show to the teacher.

2.2 Identities of Trigonometric Ratios

In a right-angled triangle, the relationships between the sides are given by Pythagoras theorem ($h^2 = p^2 + b^2$). Also, the ratios of the sides of the right angled triangle are of special importance in trigonometry.

Activity 1

Observe the given right-angled triangle. The side opposite to 90° is called the hypotenuse. The side opposite to the reference angle θ is called the perpendicular, and the side adjacent to the reference angle and to 90° is called the base.



Now, list all possible ratios that can be formed from any two sides of the right-angled triangle BNA.

Trigonometric ratios are defined as follows.

1. $\frac{\text{Oppsite side}}{\text{Hypotenuse}}$ is known as sine or shortly sin. Thus, $\sin\theta = \frac{p}{h}$

2. $\frac{\text{Adjacent side}}{\text{Hypotenuse}}$ is known as cosine or shortly cos. Thus, $\cos\theta = \frac{b}{h}$

3. $\frac{\text{Opposite side}}{\text{Adjacent side}}$ is known as tangent or shortly tan. Thus, $\tan\theta = \frac{p}{b}$

These three ratios are called basic trigonometric ratio. The reciprocal ratios of them are as follows.

4. $\frac{\text{Hypotenuse}}{\text{Oppsite side}}$ is known as cosecant or shortly cosec. Thus, $\text{cosec}\theta = \frac{h}{p}$

5. $\frac{\text{Hypotenuse}}{\text{Adjacent side}}$ is known as secant or shortly sec. Thus,$\sec\theta = \frac{h}{b}$

6. $\frac{\text{Adjacent side}}{\text{Opposite side}}$ is known as cotangent or shortly cot. Thus, $\cot\theta = \frac{b}{p}$

2.2.2 Identities of Trigonometric Ratios

Problem: a. $\sin\theta \times \text{cosec}\theta = \ldots\ldots\ldots\ldots?$ b. $\tan\theta \times \cot\theta = \ldots\ldots\ldots\ldots?$

c. $\cos\theta \times \sec\theta = \ldots\ldots\ldots\ldots?$

According to the definition, we know that, $\sin\theta = \frac{p}{h}$ and $\text{cosec}\theta = \frac{h}{p}$. Find the product of sinθ and cosecθ. Similarly do (b) and (c).

Reciprocal Relation

1. $\sin\theta = \frac{1}{\text{cosec}\theta}$
2. $\text{cosec}\theta = \frac{1}{\sin\theta}$
3. $\sin\theta \times \text{cosec}\theta = 1$
4. $\cos\theta = \frac{1}{\sec\theta}$
5. $\sec\theta = \frac{1}{\cos\theta}$
6. $\cos\theta \times \sec\theta = 1$
7. $\tan\theta = \frac{1}{\cot\theta}$
8. $\cot\theta = \frac{1}{\tan\theta}$
9. $\tan\theta \times \cot\theta = 1$

Thus, sinθ and cosecθ, tanθ and cotθ, cosθ and secθ are the reciprocal relations of each other.

Again, according to the definition

i. $\tan\theta = \frac{p}{b}$

Divide by h in both numerator and denominator, $\tan\theta = \frac{\frac{p}{h}}{\frac{b}{h}} = \frac{\sin\theta}{\cos\theta}$

or, $\frac{\sin\theta}{\cos\theta} = \frac{\frac{p}{h}}{\frac{b}{h}} = \frac{p}{b} = \tan\theta$ Thus, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\because \sin\theta = \frac{p}{h}$ and $\cos\theta = \frac{b}{h}$

ii. $\cot\theta = \frac{b}{p}$

Divide by h in both numerator and denominator, $\cot\theta = \frac{\frac{b}{h}}{\frac{p}{h}} = \frac{\cos\theta}{\sin\theta}$

Or, $\frac{\cos\theta}{\sin\theta} = \frac{\frac{b}{h}}{\frac{p}{h}} = \frac{b}{p} = \cot\theta$ Thus, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\because \cos\theta = \frac{b}{h}$ and $\sin = \frac{p}{h}$

These relations are called quotient relation.

Quotient relation

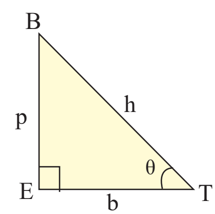
1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\cot\theta = \frac{\cos\theta}{\sin\theta}$

2.2.3 Pythagoras Relation

a. Prove that : $\sin^2\theta + \cos^2\theta = 1$

Let BET is a right angled triangle, where $\angle BET = 90°$. From the reference angle $\angle BTE = \theta$, $\sin\theta = \frac{p}{h} = \frac{BE}{BT}$ and $\cos\theta = \frac{b}{h} = \frac{ET}{BT}$.

According to Pythagoras theorem, $p^2 + b^2 = h^2$ from the figure.



Or, $BE^2 + ET^2 = BT^2$ {∵ Where p = BE, b = ET and h = BT}

Or, $\frac{BE^2 + ET^2}{BT^2} = \frac{BT^2}{BT^2}$ {∵ Dividing both sides by BT^2}

Or, $\frac{BE^2}{BT^2} + \frac{ET^2}{BT^2} = 1$ {∵ Arranging the terms}

Or, $\left(\frac{BE}{BT}\right)^2 + \left(\frac{ET}{BT}\right)^2 = 1$

Or, $(\sin\theta)^2 + (\cos\theta)^2 = 1$

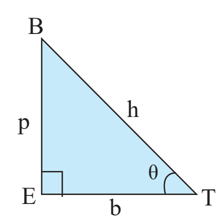
$\sin^2\theta + \cos^2\theta = 1$ *Proved.*

b. Prove that : $\sec^2\theta - \tan^2\theta = 1$

Proof: BET is a right angled triangle, where $\angle BET = 90°$.

From the reference angle $\angle BTE = \theta$, $\tan\theta = \frac{p}{b} = \frac{BE}{ET}$ and $\sec\theta = \frac{h}{b} = \frac{BT}{ET}$

In the figure, according to Pythagoras theorem, $p^2 + b^2 = h^2$



Or, $BE^2 + ET^2 = BT^2$ {∵ Where, p = BE, b = ET and h = BT}

Or, $\frac{BE^2 + ET^2}{ET^2} = \frac{BT^2}{ET^2}$ {∵ Dividing both side by ET^2}

Or, $\frac{BE^2}{ET^2} + \frac{ET^2}{ET^2} = \frac{BT^2}{ET^2}$ {∵ Arranging the terms }

Or, $\left(\frac{BE}{ET}\right)^2 + 1 = \left(\frac{BT}{ET}\right)^2$

Or, $(\tan\theta)^2 + 1 = (\sec\theta)^2$

Or, $\tan^2\theta + 1 = \sec^2\theta$

Or, $1 = \sec^2\theta - \tan^2\theta$

Hence, $\sec^2\theta - \tan^2\theta = 1$ Proved.

c. Prove that : $\text{cosec}^2\theta - \cot^2\theta = 1$

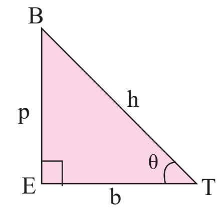
Proof: BET is a right angled triangle, where $\angle BET = 90°$. From the reference angle $\angle BTE = \theta$,

$\text{cosec}\theta = \frac{h}{p} = \frac{BT}{BE}$ and $\cot\theta = \frac{b}{p} = \frac{ET}{BE}$

In the figure, according to Pythagoras theorem, $p^2 + b^2 = h^2$

Or, $BE^2 + ET^2 = BT^2$ {∵ Where, p = BE, b = ET and h = BT}

Or, $\frac{BE^2+ET^2}{BE^2} = \frac{BT^2}{BE^2}$ {∵ Dividing both side by BE^2}



Or, $\frac{BE^2}{BE^2} + \frac{ET^2}{BE^2} = \frac{BT^2}{BE^2}$ {∵ Arranging the terms}

Or, $1 + \left(\frac{ET}{BE}\right)^2 = \left(\frac{BT}{BE}\right)^2$

Or, $1 + (\cot\theta)^2 = (\text{cosec}\,\theta)^2$

Or, $1 + \cot^2\theta = \text{cosec}^2\theta$

Or, $1 = \text{cosec}^2\theta - \cot^2\theta$

Thus, $\text{cosec}^2\theta - \cot^2\theta = 1$ Proved.

Summary of the trigonometric relations obtained from Pythagoras theorem

1. $\sin^2\theta + \cos^2\theta = 1$
   - (i) $\sin^2\theta = 1 - \cos^2\theta$ (ii) $\sin\theta = \sqrt{1 - \cos^2\theta}$
   - (iii) $\cos^2\theta = 1 - \sin^2\theta$ (iv) $\cos\theta = \sqrt{1 - \sin^2\theta}$
2. $\sec^2\theta - \tan^2\theta = 1$
   - (i) $\tan^2\theta = \sec^2\theta - 1$ (ii) $\tan\theta = \sqrt{\sec^2\theta - 1}$
   - (iii) $\sec^2\theta = 1 + \tan^2\theta$ (iv) $\sec\theta = \sqrt{1 + \tan^2\theta}$
3. $\text{cosec}^2\theta - \cot^2\theta = 1$
   - (i) $\cot^2\theta = \text{cosec}^2\theta - 1$ (ii) $\cot\theta = \sqrt{\text{cosec}^2\theta - 1}$
   - (iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$ (iv) $\text{cosec}\,\theta = \sqrt{1 + \cot^2\theta}$

Example 1

Add: sinA + cosB + 2sinA + 5cosB

Solution: Here,

sinA + cosB + 2sinA + 5cosB

= sinA + 2sinA + cosB + 5cosB {∵ Arranging like terms}

= 3sinA + 6cosB {∵ Adding like terms}

Example 2

Subtract: 7secA – 2secA

Solution: Here,

$7\sec A - 2\sec A = 5\sec A$ {∵ Subtracting like terms}

Example 3

Multiply: $(\sin A - \cos B)(\sin A + \cos B)(\sin^2 A + \cos^2 B)$

Solution: Here,

$(\sin A - \cos B)(\sin A + \cos B)(\sin^2 A + \cos^2 B)$

$= (\sin^2 A - \cos^2 B)(\sin^2 A + \cos^2 B)$ {∵ $(a + b)(a - b) = a^2 - b^2$}

$= (\sin^4 A - \cos^4 B)$ {∵ $(a + b)(a - b) = a^2 - b^2$}

Example 4

Factorize: $2\sec^2\theta + \sec\theta - 6$

Solution: Here,

$2\sec^2\theta + \sec\theta - 6 = 2\sec^2\theta + (4 - 3)\sec\theta - 6$

$= 2\sec^2\theta + 4\sec\theta - 3\sec\theta - 6$

$= 2\sec\theta(\sec\theta + 2) - 3(\sec\theta + 2)$

$= (\sec\theta + 2)(2\sec\theta - 3)$

Example 5

Prove that: $\sin^2\alpha \times \sec\alpha \times \cot^2\alpha = \cos\alpha$

Solution: Here,

L.H.S. $= \sin^2\alpha \times \sec\alpha \times \cot^2\alpha$

$= \sin^2\alpha \times \frac{1}{\cos\alpha} \times \frac{\cos^2\alpha}{\sin^2\alpha}$

$= \cos\alpha$ = R.H.S. *proved.*

Example 6

Prove that: $\sec^4\beta - \sec^2\beta = \tan^4\beta + \tan^2\beta$

Solution: Here,

L.H.S. $= \sec^4\beta - \sec^2\beta$

$= \sec^2\beta(\sec^2\beta - 1)$

$= (1 + \tan^2\beta)(1 + \tan^2\beta - 1)$ {∵ $\sec^2\beta = 1 + \tan^2\beta$}

$= (1 + \tan^2\beta)(\tan^2\beta)$

$= \tan^4\beta + \tan^2\beta$ = R.H.S. *proved.*

Example 7

Prove that: $\sqrt{1 - 2\sin\theta\cdot\cos\theta} = \sin\theta - \cos\theta$

Solution: Here,

L.H.S. $= \sqrt{1 - 2\sin\theta\cdot\cos\theta}$

$= \sqrt{\sin^2\theta + \cos^2\theta - 2\sin\theta\cdot\cos\theta}$

$= \sqrt{(\sin\theta - \cos\theta)^2}$

$= \sin\theta - \cos\theta$ = R.H.S. *proved.*

Thought Provoking Question: Is $\sqrt{1 - 2\sin\theta\cdot\cos\theta} = \cos\theta - \sin\theta$ proved? Discuss.

Exercise 2.2 (A)

1. Fill in the blanks.
   a. $\operatorname{cosec}\theta$ in terms of $\sin\theta$ =……..
   b. $\tan\theta$ in terms of $\cot\theta$ = ………
   c. $\cos\theta$ in terms of $\sec\theta$ = ……..
   d. $\cot\theta$ in terms of $\tan\theta$ = ………..
   e. $\sin\theta$ in terms of $\operatorname{cosec}\theta$ = ……..
   f. $\sec\theta$ in terms of $\cos\theta$ = ………..
   g. $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$ =…
   h. $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$ = ...
2. Write the formula related to quotient relation.
3. Write down any three trigonometric identities.
4. Prepare a list of all possible trigonometric relationships obtained from the Pythagoras theorem.
5. Simplify:
   a. $\tan\theta + 3\tan\theta$
   b. $4\cot A + 6\cot A + 2\cot A$
   c. $\sec^2 A + 10\sec^2 A$
   d. $\sin^3 x + 3\sin^3 x + 2\sin^3 x$
   e. $\sin\theta - 4\sin\theta$
   f. $7\tan A - 2\tan A$
   g. $\sec^2 A - 10\sec^2 A$
   h. $3\sin^3 x - 2\sin^3 x$
   i. $\sin\theta - 4\sin\theta + 20\sin\theta$
   j. $8\tan A - 2\tan A + 12\tan A$
   k. $9\cos^2 A - 5\cos^2 A + 16\cos^2 A$
   l. $7\operatorname{cosec}^3 x - 5\operatorname{cosec}^3 x + 12\operatorname{cosec}^3 x$
6. Multiply:
   a. $(\sin A + \sin B)(\sin A - \sin B)$
   b. $(1 - \cos\theta)(1 + \cos\theta)$
   c. $(1 + \sin\theta)(1 - \sin\theta)$
   d. $(1 + \cot^2 A)(1 + \cot^2 A)$
   e. $(1 + \sin\theta)(1 - \sin\theta)(1 + \sin^2\theta)$
   f. $(1 + \tan\theta)(1 - \tan\theta)(1 + \tan^2\theta)$

7. Factorize:

a. $\cos^2A - \sin^2A$ b. $\sec^2A - \text{cosec}^2A$ c. $\cos^2A + \sin^2A.\cos^2A$

d. $\tan^3\theta - \cot^3\theta$ e. $\sec^4\theta - \text{cosec}^4\theta$ e. $\sin^2x + 3\sin x + 2$

8. Prove that:

a. $\cot A \sin A = \cos A$ b. $\cos A\ \text{cosec}A = \cot A$

c. $\sec\theta \sin\theta \cot\theta = 1$ d. $\tan\theta \cos\theta = \sin\theta$

e. $\dfrac{\sin\theta.\text{cosec}\,\theta}{\sec\theta} = \cos\theta$ f. $\dfrac{\tan\theta.\cot\theta}{\sec\theta\ \text{cosec}\,\theta} = \sin\theta\cos\theta$

9. Prove that:

a. $(1 - \cos^2\theta)(1 + \tan^2\theta) = \tan^2\theta$ b. $(1 + \cot^2A)(1 - \sin^2A) = \cot^2A$

c. $\cos^2\theta - \cos^2\theta\,.\,\sin^2\theta = \cos^4\theta$ d. $(1 - \sin^2A)\,\text{cosec}^2A = \cot^2A$

e. $\sin^2\theta + \sin^2\theta\,.\,\cot^2\theta = 1$ f. $\sin\theta(1 + \cot^2\theta) = \text{cosec}\,\theta$

g. $\cos A(1 + \tan^2A) = \sec A$ h. $(\sin x - \cos x)^2 = 1 - 2\sin x\cos x$

10. Prove that:

a. $\sin^2A - \cos^2B = \sin^2B - \cos^2A$ b. $\cot^2B - \cos^2B = \cot^2B\cos^2B$

c. $\tan^2C - \sin^2C = \sin^2C\tan^2C$ d. $\sqrt{1 + 2\sin\alpha\cdot\cos\alpha}) = \cos\alpha + \sin\alpha$

e. $\cos^2\theta\sqrt{1 + \cot^2\theta} \times \sqrt{\text{cosec}^2\theta} = \cot^2\theta$ f. $\cos\theta.\text{cosec}\,\theta\sqrt{(\sec^2\theta - 1)} = 1$

g. $\dfrac{\cot\theta}{\sqrt{1 + \cot^2\theta}} = \cos\theta$ h. $\dfrac{\sqrt{\sec^2\theta - 1}}{\sec\theta} = \sin\theta$

Answer

1 - 4. Show to the teacher. 5. a. $4\tan\theta$ b. $12\cot A$ c. $11\sec^2A$ d. $6\sin^3x$ e. $-3\sin\theta$ f. $5\tan A$ g. $-9\sec^2A$ h. $\sin^3x$ i. $17\sin\theta$ j. $18\tan A$ k. $20\cos^2A$ *l.* 14cose^3x 6. a. $\sin^2A - \sin^2B$ b. $1 - \cos^2\theta$ c. $1 - \sin^2\theta$ d. $1 + 2\cot^2A + \cot^4A$ e. $1 - \sin^4\theta$ f. $1 - \tan^4\theta$

7. a. $(\cos A + \sin A)(\cos A - \sin A)$ b. $(\sec A + \text{cosec}A)(\sec A - \text{cosec}A)$ c. $\cos^2A(1 + \sin^2A)$ d. $(\tan\theta - \cot\theta)(\tan^2\theta + \tan\theta\cot\theta + \cot^2\theta)$ e. $(\sec^2\theta + \text{cosec}^2\theta)(\sec\theta + \text{cosec}\theta)(\sec\theta - \text{cosec}\theta)$ f. $(\sin x + 2)(\sin x + 1)$ 8 to 10. Show to the teacher.

2.2.4 Typical Patterns of Identities in Trigonometry

In trigonometry, some identities are not in general form but are different forms. How can these identities prove? Does proving them require different knowledge? Study the examples given below and discuss.

Example 1

Prove that: $\frac{1}{\sec^2 A} + \frac{1}{\csc^2 A} = 1$

Solution: Here,

L.H.S. = $\frac{1}{\sec^2 A} + \frac{1}{\csc^2 A}$

$= \cos^2 A + \sin^2 A$ {∵ According to reciprocal formula, $\frac{1}{\sec^2 A} = \cos^2 A$ and $\frac{1}{\csc^2 A} = \sin^2 A$}

$= 1$ = R.H.S. *proved.*

Example 2

Prove that: $\frac{1}{\sec A - \tan A} = \sec A + \tan A = \frac{1+\sin A}{\cos A}$

Solution: Here,

L.H.S. = $\frac{1}{\sec A - \tan A}$

Multiply by (secA + tanA) in numerator and denominator

$= \frac{1}{\sec A - \tan A} \times \frac{\sec A + \tan A}{\sec A + \tan A}$

$= \frac{\sec A + \tan A}{\sec^2 A - \tan^2 A}$ [∵ $(a + b)(a - b) = (a^2 - b^2)$]

$= \frac{\sec A + \tan A}{1}$ [∵ Using the Pythagoras relation in trigonometry $\sec^2 A - \tan^2 A = 1$]

$= \sec A + \tan A$

Again, $\sec A + \tan A$

$= \frac{1}{\cos A} + \frac{\sin A}{\cos A}$

$= \frac{1 + \sin A}{\cos A}$

= R.H.S.

Thus, L.H.S. = R.H.S. *proved.*

Example 3

Prove that: $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

> Multiply by 1 + sinθ in numerator and denominator under the terms of first square root and multiply by 1 – sinθ in numerator and denominator under the terms of second square root.

Solution: Here,

$$\text{L.H.S} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)}{(1-\sin\theta)} \times \frac{(1+\sin\theta)}{(1+\sin\theta)}} + \sqrt{\frac{(1-\sin\theta)}{(1+\sin\theta)} \times \frac{(1-\sin\theta)}{(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} \quad (\because 1 - \sin^2\theta = \cos^2\theta)$$

$$= \frac{1+\sin\theta}{\cos\theta} + \frac{1-\sin\theta}{\cos\theta} = \frac{1+\sin\theta+1-\sin\theta}{\cos\theta} = \frac{2}{\cos\theta}$$

$= 2\sec\theta \quad$ = R.H.S. *Proved.*

Example 4

Prove that: $\dfrac{1-\cos\theta}{1+\cos\theta} = (\cot\theta - \text{cosec}\theta)^2$

Solution: Here,

> Multiply by 1 – cosθ in numerator and denominator

$$\text{L.H.S} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\theta}{1+\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta}$$

$$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{(1-\cos\theta)^2}{\sin^2\theta} = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$

$$= (\text{cosec}\theta - \cot\theta)^2$$

$$= [-(\cot\theta - \text{cosec}\theta)]^2$$

$$= (\cot\theta - \text{cosec}\theta)^2$$

= R.H.S, *Proved.*

Example 5

Prove that: $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

Solution: Here,

$$\text{L.H.S} = \frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A}$$

$$= \frac{\cos A}{1-\frac{\sin A}{\cos A}} + \frac{\sin A}{1-\frac{\cos A}{\sin A}}$$

$$= \frac{\cos A}{\frac{\cos A-\sin A}{\cos A}} + \frac{\sin A}{\frac{\sin A-\cos A}{\sin A}}$$

$$= \frac{\cos^2 A}{\cos A-\sin A} - \frac{\sin^2 A}{\cos A-\sin A}$$

$$= \frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}$$

$$= \frac{(\cos A-\sin A)(\cos A+\sin A)}{\cos A-\sin A}$$

$$= \sin A + \cos A$$

$= \text{R.H.S}$. *proved.*

Example 6

Prove that: $\frac{1-\sin^4 A}{\cos^4 A} = 1 + 2\tan^2 A$

Solution: Here,

$$\text{L.H.S} = \frac{1-\sin^4 A}{\cos^4 A}$$

$$= \frac{1^2-(\sin^2 A)^2}{\cos^2 A.\cos^2 A}$$

$$= \frac{(1-\sin^2 A)(1+\sin^2 A)}{\cos^2 A.\cos^2 A}$$

$$= \frac{(\cos^2 A)(1+\sin^2 A)}{\cos^2 A.\cos^2 A}$$

$$= \frac{(1+\sin^2 A)}{\cos^2 A}$$

$$= \frac{1}{\cos^2 A} + \frac{\sin^2 A}{\text{Sin}^2 A}$$

$$= \sec^2 A + \tan^2 A$$

$$= 1 + \tan^2 A + \tan^2 A$$

$$= 1 + 2\tan^2 A$$

$= \text{R.H.S}$. *proved.*

Example 7

Prove that: $\sin^6A - \cos^6A = (2\sin^2A - 1)(\cos^2A + \sin^4A)$

Solution: Here,

$$\text{L.H.S} = \sin^6A - \cos^6A$$
$$= (\sin^2 A)^3 - (\cos^2 A)^3$$
$$= (\sin^2A - \cos^2A)\ \{(\sin^2 A)^2 + \sin^2A\cos^2A + (\cos^2 A)^2\}$$
$$= \{\sin^2A - (1 - \sin^2A)\}\ (\sin^4A + \sin^2A\cos^2A + \cos^4A)$$
$$= \{\sin^2A - 1 + \sin^2A)\}\ \{\sin^4A + \cos^2A\,(\sin^2A + \cos^2A)\}$$
$$= (2\sin^2A - 1)\ \{\sin^4A + \cos^2A\,(1)\}$$
$$= (2\sin^2A - 1)\ (\cos^2A + \sin^4A)$$

= R.H.S. *proved.*

Using of algebraic formula
$(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Example 8

Prove that: $\dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \dfrac{\sin x + 1}{\cos x}$

Solution: Here,

$$\text{L.H.S} = \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$$
$$= \frac{\tan x + \sec x - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$$
$$= \frac{\tan x + \sec x - (\sec x - \tan x)(\sec x + \tan x)}{\tan x - \sec x + 1}$$
$$= \frac{(\tan x + \sec x)(1 - \sec x + \tan x)}{\tan x - \sec x + 1}$$
$$= \tan x + \sec x$$
$$= \frac{\sin x}{\cos x} + \frac{1}{\cos x}$$
$$= \frac{\sin x + 1}{\cos x}$$

= R.H.S. *proved.*

Thought Provoking Question:
Why is $1 = \sin^2 x + \cos^2 x$ not used?

Example 9

Prove that: $\dfrac{\sin A}{\sec A + \tan A - 1} + \dfrac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$

Solution: Here,

$$\text{L.H.S} = \frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec}A + \cot A - 1}$$

$$= \frac{\sin A\,(\operatorname{cosec}A + \cot A - 1) + \cos A\,(\sec A + \tan A - 1)}{(\sec A + \tan A - 1)(\operatorname{cose}A + \cot A - 1)}$$

$$= \frac{\sin A\operatorname{cosec}A + \sin A\cot A - \sin A + \cos A\sec A + \cos A\tan A - \cos A)}{(\sec A + \tan A - 1)(\operatorname{cose}A + \cot A - 1)}$$

$$= \frac{1 + \cos A - \sin A + 1 + \sin A - \cos A}{\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A} - 1\right)\left(\frac{1}{\sin A} + \frac{\cos A}{\sin A} - 1\right)}$$

$$= \frac{2}{\left(\frac{1 + \sin A - \cos A}{\cos A}\right)\left(\frac{1 + \cos A - \sin A}{\sin A}\right)}$$

$$= \frac{2}{1 + (\sin A - \cos A)\{(1 - (\sin A - \cos A)\}} \times \frac{\sin A.\cos A}{1}$$

$$= \frac{2\sin A.\cos A}{(1)^2 - (\sin A - \cos A)^2}$$

$$= \frac{2\sin A.\cos A}{1 - (\sin^2 A - 2\sin A\cos A + \cos^2 A)}$$

$$= \frac{2\sin A.\cos A}{1 - (1 - 2\sin A.\cos A)}$$

$$= \frac{2\sin A.\cos A}{1 - 1 + 2\sin A.\cos A}$$

$$= \frac{2\sin A.\cos A}{2\sin A.\cos A}$$

$= 1$

$=$ R.H.S. *proved.*

cosA tanA = sinA
sinA cosecA = 1
sinA cotA = cosA
CosA secA = sinA

Example 10

Prove that: $(5\cot^2\theta + 1)(2 - \sin^2\theta) = (5 - 4\sin^2\theta)(1 + 2\cot^2\theta)$

Solution: Here,

$$\text{L.H.S} = (5\cot^2\theta + 1)(2 - \sin^2\theta)$$

$$= \left(5\frac{\cos^2\theta}{\sin^2\theta} + 1\right)(2 - \sin^2\theta)$$

$$= \left(\frac{5\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)(2 - \sin^2\theta)$$

$$= \{5(1 - \sin^2\theta) + \sin^2\theta\}\left(\frac{2 - \sin^2\theta}{\sin^2\theta}\right)$$

$$= (5 - 5\sin^2\theta + \sin^2\theta)\left(\frac{2}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}\right)$$

$$= (5 - 4\sin^2\theta)(2\operatorname{cosec}^2\theta - 1)$$

$$= (5 - 4\sin^2\theta)\{2(1 + \cot^2\theta) - 1\}$$

$$= (5 - 4\sin^2\theta)(2 + 2\cot^2\theta - 1)$$

$$= (5 - 4\sin^2\theta)(1 + 2\cot^2\theta)$$

= R.H.S. *proved.*

Example 11

Prove that: $\frac{1}{\sec B + \tan B} + \frac{1}{\sec B - \tan B} = \frac{1}{\cos B} + \frac{1}{\cos B}$

Or, $\frac{1}{\sec B + \tan B} - \frac{1}{\cos B} = \frac{1}{\cos B} - \frac{1}{\sec B - \tan B}$

Solution: Here,

$$\text{L.H.S} = \frac{1}{\sec B + \tan B} + \frac{1}{\sec B - \tan B}$$

$$= \frac{\sec B - \tan B + \sec B + \tan B}{(\sec B + \tan B)(\sec B - \tan B)}$$

$$= \frac{2\sec B}{1}$$

$$= 2\sec B$$

$$= \frac{2}{\cos B} = \frac{1}{\cos B} + \frac{1}{\cos B}$$

= R.H.S. *proved.*

Alternative Method

$$\text{L.H.S.} = \frac{1}{\sec B + \tan B} - \frac{1}{\cos B}$$

$$= \frac{\sec^2 B - \tan^2 B}{\sec B + \tan B} - \frac{1}{\cos B}$$

$$= \frac{(\sec B + \tan B)(\sec B - \tan B)}{\sec B + \tan B} - \frac{1}{\cos B}$$

$$= \sec B - (\tan B + \sec B)$$

$$= \frac{1}{\cos B} \square \frac{(\tan B + \sec B)}{1} = \frac{1}{\cos B} - \frac{(\tan B + \sec B)}{\sec^2 B - \tan^2 B}$$

$$= \frac{1}{\cos B} - \frac{(\tan B + \sec B)}{(\tan B + \sec B)(\tan B - \sec B)}$$

$$= \frac{1}{\cos B} - \frac{1}{\sec B - \tan B}$$ = R.H.S. *proved.*

Exercise 2.2 (B)

1. Prove that:

a. $\dfrac{1}{\cos^2 A} - \dfrac{1}{\cot^2 A} = 1$

b. $\dfrac{1}{\text{cosec}^2 A} + \dfrac{1}{\sec^2 A} = 1$

c. $\dfrac{1}{\sin^2 A} - \dfrac{1}{\tan^2 A} = 1$

d. $\dfrac{\sec A}{\cos A} - \dfrac{\tan A}{\cot A} = 1$

e. $\dfrac{1 - \tan A}{1 + \tan A} = \dfrac{\cot A - 1}{\cot A + 1}$

f. $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \dfrac{\cot^2 A - 1}{\cot^2 A + 1}$

g. $\dfrac{\sin^3\alpha + \cos^3\alpha}{\cos\alpha + \sin\alpha} = 1 - \sin\alpha\cos\alpha$

h. $\dfrac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha} = \dfrac{1 + \tan\alpha}{1 - \tan\alpha}$

i. $\dfrac{\cot^2\beta}{1 + \cot^2\beta} = \cos^2\beta$

j. $\dfrac{\tan^2 A - \cot^2 A}{1 + \cot^2 A} = \tan^2 A - 1$

k. $\dfrac{\cos^2 A - \sin^2 A}{\sin A\cos^2 A - \cos A\sin^2 A} = \text{cosec}A + \sec A$

l. $\dfrac{\sin A + \sin B}{\cos A + \cos B} + \dfrac{\cos A - \cos B}{\sin A - \sin B} = 0$

2. Prove that:

a. $\dfrac{1}{\text{cosec}\theta + \cot\theta} = \text{cosec}\theta - \cot\theta = \dfrac{1 - \cos\theta}{\sin\theta}$

b. $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta = \dfrac{1 - \sin\theta}{\cos\theta}$

c. $\dfrac{1}{\text{cosec}\alpha - \cot\alpha} = \dfrac{1 + \cos\alpha}{\sin\alpha} = \text{cosec}\alpha + \cot\alpha$

d. $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta = \dfrac{1 + \sin\theta}{\cos\theta}$

3. Prove that:

a. $\sqrt{\dfrac{1 + \sin\theta}{1 - \sin\theta}} = \sec\theta + \tan\theta$

b. $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} = \sec\theta - \tan\theta$

c. $\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \text{cosec}\theta - \cot\theta$

d. $\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{\sin A}{1 + \cos A}$

e. $\sqrt{\dfrac{1 + \cos\theta}{1 - \cos\theta}} = \text{cosec}\theta + \cot\theta$

f. $\sqrt{\dfrac{1 + \tan^2 A}{1 + \cot^2 A}} = \tan A$

4. Prove that: :

a. $\frac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$ b. $\frac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

c. $\frac{1-\cos x}{1+\cos x} = (\mathrm{cosec}\,x - \cot x)^2$ d. $(\mathrm{cosec}\,x + \cot x)^2 = \frac{1+\cos x}{1-\cos x}$

5. Prove that:

a. $\frac{1}{1+\cos\alpha} + \frac{1}{1-\cos\alpha} = 2\mathrm{cosec}^2\alpha$

b. $\frac{1}{1-\sin A} - \frac{1}{1+\sin A} = 2\sin A.\sec^2 A$

c. $\frac{1}{1-\cos A} - \frac{1}{1+\cos A} = 2\cot A.\mathrm{cosec}A$

d. $\frac{\cos A}{1-\sin A} + \frac{\cos A}{1+\sin A} = 2\sec A$

e. $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\mathrm{cosec}A$

f. $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = 1 + \sec A.\mathrm{cosec}A$

g. $\frac{\tan^2 A}{\tan A - 1} - \frac{\cot A}{1-\tan A} = 1 + \sec A\mathrm{cosec}A$

h. $\frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{2}{\sin^2 A - \cos^2 A}$

6. Prove that:

a. $\frac{1-\cos^4 A}{\sin^4 A} = 2\mathrm{cosec}^2 A - 1$ b. $\frac{1-\cos^4 A}{\sin^4 A} = 1 + 2\cot^2 A$

c. $2\tan^2 x + 1 = \frac{1-\sin^4 x}{\cos^4 x}$ d. $\sec^4 A + \tan^4 A = 1 + \frac{2\tan^2 A}{\cos^2 A}$

7. Prove that:

a. $(\sin\theta + \cos\theta)^3 = 3(\sin\theta + \cos\theta) - 2(\sin^3\theta + \cos^3\theta)$

b. $(1 + \sin\theta + \cos\theta)^2 = 2(1 + \sin\theta)(1 + \cos\theta)$

c. $\sec^6 A - \tan^6 A = 1 + 3\tan^2 A.\sec^2 A$

d. $\sin^8 A - \cos^8 A = (2\sin^2 A - 1)(1 - 2\sin^2 A.\cos^2 A)$

8. Prove that:

a. $\frac{\cos A - \sin A + 1}{\cos A + \sin A + 1} = \frac{1-\sin A}{\cos A}$ b. $\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \frac{1+\cos A}{\sin A}$

c. $\frac{\text{cosecA} + \text{cotA} - 1}{1 - \text{cosecA} + \text{cotA}} = \frac{1 + \text{cosA}}{\text{sinA}} = \text{cosecA} + \text{cotA}$

d. $\frac{1 + \text{cosecA} + \text{cotA}}{1 + \text{cosecA} - \text{cotA}} = \frac{\text{cosecA} + \text{cotA} - 1}{\text{cotA} - \text{cosecA} + 1}$

e. $\frac{1 - \text{secA} + \text{tanA}}{1 + \text{secA} - \text{tanA}} = \frac{\text{secA} + \tan A - 1}{\text{secA} + \tan A + 1}$

f. $\frac{\sin\theta + \cos\theta + 1}{\sin\theta + \cos\theta - 1} - \frac{1 + \sin\theta - \cos\theta}{1 - \sin\theta + \cos\theta} = 2(1 + \text{cosec}\theta)$

9. Prove that:

a. $(\sec\theta + \tan\theta - 1)(\sec\theta - \tan\theta + 1) = 2\tan\theta$

b. $(\sec\theta - \tan\theta)\left(\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1}\right) = 1$

c. $(1 + \text{cotA} - \text{cosecA})(1 + \text{tanA} + \text{secA}) = 2$

d. $\sec^2A.\text{cosec}^2A - \tan^2A - \cot^2A = 2$

10. Prove that:

a. $(2 - \cos^2A)(1 + 2\cot^2A) = (2 + \cot^2A)(2 - \sin^2A)$

b. $(3 - 4\sin^2A)(1 - 3\tan^2A) = (3 - \tan^2A)(4\cos^2A - 3)$

c. $(3 - 4\cos^2A)(\text{cosec}^2A - 4\cot^2A) = (3 - \cot^2A)(1 - 4\cos^2A)$

11. Prove that:

a. $\frac{1}{\text{cosecA} - \text{cotA}} - \frac{1}{\text{sinA}} = \frac{1}{\text{sinA}} - \frac{1}{\text{cosecA} + \text{cotA}}$

b. $\frac{\text{cosA}}{\text{sinA} + \text{cosB}} + \frac{\text{cosB}}{\text{sinB} - \text{cosA}} = \frac{\text{cosA}}{\text{sinA} - \text{cosB}} + \frac{\text{cosB}}{\text{sinB} + \text{cosA}}$

2.3 Conversion of Trigonometric Ratios

What is meant by conversion? Is it possible to convert one trigonometric ratio into another? If yes, how can it be done? Does converting a trigonometric ratio change the angle or not? Conversion means change. Using the basic trigonometric ratios and various identities, one trigonometric ratio can be converted into another ratio. However, the conversion must be done without changing the angle.

For example: Identities: $\sin^2\theta + \cos^2\theta = 1$....... (i)

$\sec^2\theta - \tan^2\theta = 1$....... (ii)

$\text{cosec}^2\theta - \cot^2\theta = 1$....... (ii) etc.

Trigonometric ratios can be converted using the method given below.

a. By using basic trigonometric relation

b. By using Pythagoras theorem

Example 1

Express all trigonometric ratios in terms of following taking as a reference angle θ

a. Convert into sinθ by using basic trigonometric relations.

b. Convert into sinθ by using the Pythagoras theorem.

Solution: Here,

a. By using basic trigonometric relations

$\sin\theta = \sin\theta$ ……… (i)

$\text{cosec}\,\theta = \frac{1}{\sin\theta}$ ……… (ii) [relation of reciprocal]

$\cos\theta = \sqrt{1-\sin^2\theta}$ ……… (iii) [Pythagorean relation]

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}}$ ……… (iv) [Reciprocal and Pythagorean relation]

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ ……… (v) [Quotient and Pythagorean relation]

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$ ……… (vi) [Quotient and Pythagorean relation]

b. By using Pythagoras theorem

A triangle ABC is a right angled triangle, where ∠ABC = 90°. Reference angle ∠CAB = θ and let sinθ = K

We know that, $\sin\theta = \frac{p}{h} = \frac{BC}{AC}$ therefore,

$\sin\theta = K$

$\frac{BC}{AC} = \frac{K}{1} = \frac{\sin A}{1}$

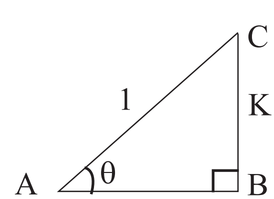
If (p) = BC = K = sinA, then h = AC = 1 and b = ?

According to Pythagoras theorem, $AC^2 = BC^2 + AB^2$ ($\because h^2 = p^2 + b^2$)

$AB^2 = AC^2 - AB^2$

$AB^2 = 1 - K^2$

$\therefore AB = \sqrt{1-K^2}$



Therefore, $\cos\theta = \frac{b}{h} = \frac{AB}{AC} = \frac{\sqrt{1-K^2}}{1} = \sqrt{1-\sin^2\theta}$

$\tan\theta = \frac{p}{b} = \frac{BC}{AB} = \frac{K}{\sqrt{1-K^2}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

$\cot\theta = \frac{b}{p} = \frac{AB}{BC} = \frac{\sqrt{1-K^2}}{K} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$

$\sec\theta = \frac{h}{b} = \frac{AC}{AB} = \frac{1}{\sqrt{1-K^2}} = \frac{1}{\sqrt{1-\sin^2\theta}}$

$\text{cosec}\,\theta = \frac{h}{p} = \frac{AC}{BC} = \frac{1}{K} = \frac{1}{\sin\theta}$

Example 2

If $\cos\theta = \frac{4}{5}$

a. What are the other trigonometric ratios? Write it.

b. Find the value of all other trigonometric ratios.

Solution: Here,

$\cos\theta = \frac{4}{5}$

a. The other trigonometric ratios are sine, tangent, cosecant, secant and cotangent.

b. Value of other trigonometric ratios

Use from basic trigonometric relation	Use of Pythagoras Theorem (Alternative method)
Here, $\cos\theta = \frac{4}{5}$ $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2}$ $= \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$, $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$ $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$, $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$	$\cos\theta = \frac{b}{h} = \frac{4}{5} = \frac{4k}{5k}$ If b = 4k and h = 5 k then to find the value of p, according to Pythagoras theorem $p = \sqrt{h^2 - b^2}$ $p = \sqrt{(5k^2) - (4k^2)}$ $p = \sqrt{25k^2 - 16k^2} = \sqrt{9k^2} = 3k$ $\sin\theta = \frac{p}{h} = \frac{3k}{5k} = \frac{3}{5}$, $\tan\theta = \frac{p}{b} = \frac{3k}{4k} = \frac{3}{4}$, $\text{cosec}\theta = \frac{h}{p} = \frac{5k}{3k} = \frac{5}{3}$, $\sec\theta = \frac{h}{b} = \frac{5k}{4k} = \frac{5}{4}$, ₹ $\cot\theta = \frac{b}{p} = \frac{4k}{3k} = \frac{4}{3}$

Example 3

If, $\tan\theta = \frac{4}{5}$ then prove: $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} = \frac{5}{14}$

Solution: Here,

$\tan\theta = \frac{4}{5}$

To Prove: $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} = \frac{5}{14}$

Use of basic trigonometric relation	Use of Pythagoras Theorem (Alternative method)
L.H.S $= \frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$ $= \frac{\frac{5\sin\theta - 3\cos\theta}{\cos\theta}}{\frac{\sin\theta + 2\cos\theta}{\cos\theta}}$ [Dividing by cosθ in numerator and denominator] $= \frac{5\frac{\sin\theta}{\cos\theta} - 3\frac{\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} + 2\frac{\cos\theta}{\cos\theta}}$ $= \frac{5\tan\theta - 3}{\tan\theta + 2}$ $= \frac{5 \times \frac{4}{5} - 3}{\frac{4}{5} + 2}$ $= \frac{\frac{20 - 15}{5}}{\frac{4 + 10}{5}}$ $= \frac{5}{14}$ = R.H.S. *proved.*	Here, $\tan\theta = \frac{4}{5}$ $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} = \frac{5}{14}$ $\tan\theta = \frac{4}{5} = \frac{p}{b}$ $\frac{p}{b} = \frac{4}{5}$ Here, $p = 4k$, $b = 5k$ Hence, $h = \sqrt{p^2 + b^2} = \sqrt{(4k)^2 + (5k)^2}$ $= \sqrt{16k^2 + 25k^2} = \sqrt{41k^2} = k\sqrt{41}$ L.H.S: $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} = \frac{5 \times \frac{p}{h} - 3 \times \frac{b}{h}}{\frac{p}{h} + 2 \times \frac{b}{h}} = \frac{\frac{5p - 3b}{h}}{\frac{p + 2b}{h}}$ $= \frac{5p - 3b}{p + 2b}$ $= \frac{5 \times 4k - 3 \times 5k}{4k + 2 \times 5k} = \frac{5k}{14k} = \frac{5}{14}$ = R.H.S. *proved.*

Example 4

If $\sin A = \frac{3}{5}$ and $\sin B = \frac{12}{13}$ prove that: $\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\frac{63}{16}$

Solution: Here, $\sin A = \frac{3}{5}$ र $\sin B = \frac{12}{13}$

To prove: $\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-63}{16}$

Use from basic trigonometric relation, We know that

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25 - 9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169 - 144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

Again, $\tan A = \frac{\sin A}{\cos A} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$, $\tan B = \frac{\sin B}{\cos B} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}$

L.H.S. = $\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \times \frac{12}{5}}$

$= \frac{\frac{3 \times 5 + 12 \times 4}{20}}{\frac{20 - 3 \times 12}{20}}$

$= \frac{15+48}{20-36} = \frac{63}{-16} = -\frac{63}{16}$

Thought Provoking Question: Can this problem be solved by using Pythagoras theorem? Solve and compare the results.

Example 5

If $\cos A = \frac{p^2 - q^2}{p^2 + q^2}$ then, find the value of tanA and cosec A.

Solution: Here, $\cos A = \frac{p^2 - q^2}{p^2 + q^2}$

tanA =? and cosecA = ?

$\cos A = \frac{p^2 - q^2}{p^2 + q^2} = \frac{b}{h}$

If $b = p^2 - q^2$ and $h = p^2 + q^2$ then $p = ?$

According to Pythagoras theorem, $p = \sqrt{h^2 - b^2} = \sqrt{(p^2 + q^2)^2 - (p^2 - q^2)^2}$

Or, $p = \sqrt{p^4 + 2.p^2.q^2 + q^4 - (p^4 - 2.p^2.q^2 + q^4)}$

Or, $p = \sqrt{p^4 + 2.p^2.q^2 + q^4 - p^4 + 2.p^2.q^2 - q^4}$

Or, $p = \sqrt{4p^2q^2} = 2pq$

$\tan A = \frac{p}{b} = \frac{2pq}{p^2 - q^2}$ and $\operatorname{cosec} A = \frac{p}{b} = \frac{p^2 + q^2}{2pq}$

Example 6

If $5\cos\theta + 12\sin\theta = 13$ prove that: $\tan\theta = \frac{12}{5}$

Solution: Here, $5\cos\theta + 12\sin\theta = 13$

To prove: $\tan\theta = \frac{12}{5}$

Use of basic trigonometric relation	Use of Pythagoras heorem (Alternative method)
Here, $5\cos\theta + 12\sin\theta = 13$ Dividing on both side by $\cos\theta$ Or, $5\frac{\cos\theta}{\cos\theta} + 12\frac{\sin\theta}{\cos\theta} = \frac{13}{\cos\theta}$ Or, $5 + 12\tan\theta = 13\sec\theta$ Squaring on both side $(5 + 12\tan\theta)^2 = (13\sec\theta)^2$ Or, $25 + 120\tan\theta + 144\tan^2\theta = 169\sec^2\theta$ Or, $25 + 120\tan\theta + 144\tan^2\theta = 169(1 + \tan^2\theta)$ Or, $25 + 120\tan\theta + 144\tan^2\theta = 169 + 169\tan^2\theta$ Or, $0 = 169\tan^2\theta - 144\tan^2\theta - 120\tan\theta + 169 - 25$ Or, $25\tan^2\theta - 120\tan\theta + 144 = 0$ Or, $(5\tan\theta)^2 - 2.5\tan\theta.12 + (12)^2 = 0$ Or, $(5\tan\theta - 12)^2 = 0$ Or, $5\tan\theta - 12 = 0$ Or, $\tan\theta = \frac{12}{5}$ *Hence, proved.*	Here, $5\cos\theta + 12\sin\theta = 13$ Or, $5\frac{b}{h} + 12\frac{p}{h} = 13$ Or, $\frac{5b + 12p}{h} = 13$ Or, $5b + 12p = 13h$ Squaring both side $(5b + 12p)^2 = (13h)^2$ Or, $25b^2 + 120pb + 144p^2 = 169(p^2 + b^2)$ Or, $0 = 169p^2 - 144p^2 - 120pb + 169b^2 - 25b^2$ Or, $25p^2 - 120pb + 144b^2 = 0$ Or, $(5p)^2 - 2.5p.12b + (12b)^2 = 0$ Or, $(5p - 12b)^2 = 0$ Or, $5p - 12b = 0$ Or, $\frac{p}{b} = \frac{12}{5}$ Or, $\tan\theta = \frac{12}{5}$ *Hence, proved.*

Exercise 2.3

1. What is meant by the conversion of trigonometric ratios?
2. a. Convert all trigonometric ratios into $\cos\theta$ with respect to the reference angle θ.

   b. Convert all trigonometric ratios into $\tan\alpha$ with respect to the reference angle α.

   c. Convert all trigonometric ratios into $\sec\theta$ with respect to the reference angle θ.

   d. Convert all trigonometric ratios into $\cot\alpha$ with respect to the reference angle α.
3. Which of the following is correct when converting cotA into sinA?

   a. $\frac{\sin A}{\sqrt{1-\sin^2 A}}$ b. $\frac{1}{\sqrt{1-\sin^2 A}}$ c. $\frac{\sqrt{1-\sin^2 A}}{\sin A}$ d. $\frac{1}{\sin A}$
4. Which of the following is correct when converting secA into cotA?

   a. $\frac{\sqrt{1+\cot^2 A}}{\cot A}$ b. $\frac{\cot A}{\sqrt{1+\cot^2 A}}$ c. $\frac{1}{\cot A}$ d. $\frac{1-\cot^2 A}{1+\cot^2 A}$
5. If $\sin\theta = \frac{3}{5}$ then,

   a. What are the other trigonometric ratios? Write it.

   b. Find the values of other trigonometric ratios.

6. a. If $\tan\theta = \frac{4}{5}$, find the values of other trigonometric ratios.

b. If $\cot\theta = \frac{1}{\sqrt{3}}$, find the values of other trigonometric ratios.

c. If $\cos\alpha = \frac{24}{25}$, find the values of other trigonometric ratios.

d. If $\text{cosec}A = \sqrt{2}$, find the values of other trigonometric ratios.

7. a. If $\tan\theta = \frac{2}{3}$, find the value of $\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$.

b. If $\cot\theta = \frac{4}{3}$, find the value of $\frac{3\sin\theta - 2\cos\theta}{2\sin\theta + 3\cos\theta}$.

c. If $\sin\theta = \frac{3}{5}$, find the value of $5\cos\theta + 4\tan\theta$.

d. If $\cos\theta = \frac{4}{5}$, find the value of $5\sin\theta + 3\cot\theta$.

e. If $\cot\theta = \frac{12}{5}$, find the value of $13\cos\theta + 24\tan\theta$.

8. If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, find the value of the following trigonometric expressions.

a. $\sin A \cos B + \cos A \sin B$

b. $\cos A \cos B + \sin A \sin B$

c. $\sin A \cos B - \cos A \sin B$

d. $\cos A \cos B - \sin A \sin B$

e. $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

f. $\frac{\tan A - \tan B}{1 + \tan A \text{tab} B}$

9. If $(m^2 + n^2) \cos A = m^2 - n^2$ then,

a. Find the value of cosA.

b. Find the values of sinA and tanA.

10. a. If $\cot\beta = \frac{p}{q}$, then prove that: $\frac{p\cos\beta - q\sin\beta}{p\cos\beta + q\sin\beta} = \frac{p^2 - q^2}{p^2 + q^2}$

b. If $\sin\theta = \frac{a}{b}$, then prove that: $\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$

c. If $\tan\alpha = \frac{2xy}{x^2 - y^2}$, then prove that: $\sin\alpha = \frac{2xy}{x^2 + y^2}$

11. a. If $4\cos\theta + 3\sin\theta = 5$, then prove that: $\tan\theta = \frac{3}{4}$

b. If $12\sin\theta = 13 - 5\cos\theta$, then find the values of $\tan\theta$ and $\cot\theta$

c. If $8\sin\theta - \cos\theta - 4 = 0$, then prove that: $\cot\theta = \frac{4}{3}$

d. If $3\sin A + 4\cos A = 5$, then prove that: $\cos A = \frac{4}{5}$

Answer

1. Show to the teacher.

2. a. $\sin\theta = \sqrt{1-\cos^2\theta}$, $\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$, $\text{cosec}\theta = \frac{1}{\sqrt{1-\cos^2\theta}}$, $\sec\theta = \frac{1}{\cos\theta}$

   b. $\cot\alpha = \frac{1}{\tan\alpha}$, $\sec\alpha = \sqrt{1+\tan^2\alpha}$, $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$, $\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$, $\text{cosec}\alpha = \frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$

   c. $\cos\theta = \frac{1}{\sec\theta}$, $\sin\theta = \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$ $\tan\theta = \sqrt{\sec^2\theta-1}$, $\cot\theta = \frac{1}{\sqrt{\sec^2\theta-1}}$, $\text{cosec}\theta = \frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$,

   d. $\tan\alpha = \frac{1}{\cot\alpha}$, $\text{cosec}\alpha = \sqrt{1+\cot^2\alpha}$, $\sin\alpha = \frac{1}{\sqrt{1+\cot^2\alpha}}$, $\cos\alpha = \frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$, $\sec\alpha = \frac{\sqrt{1+\cot^2\alpha}}{\cot\alpha}$

3. c 4. b

5. a. cosine, tangent, cosec, secant and cotangent

   b. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\text{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

6. a. $\sin\theta = \frac{4}{\sqrt{41}}$, $\cos\theta = \frac{5}{\sqrt{41}}$, $\text{cosec}\theta = \frac{\sqrt{41}}{4}$, $\sec\theta = \frac{\sqrt{41}}{5}$, $\cot\theta = \frac{5}{4}$

   b. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\text{cosec}\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = 2$

   c. $\sin\alpha = \frac{7}{25}$, $\tan\alpha = \frac{7}{24}$, $\text{cosec}\alpha = \frac{25}{7}$, $\sec\alpha = \frac{25}{24}$, $\cot\alpha = \frac{24}{7}$

   d. $\sin A = \frac{1}{\sqrt{2}}$, $\cos A = \frac{1}{\sqrt{2}}$, $\tan A = 1$, $\sec A = \sqrt{2}$ and $\cot A = 1$

7. a. $-\frac{1}{5}$ b. $\frac{1}{18}$ c. 7 d. 7 e. 22

8. a. $\frac{63}{65}$ b. $\frac{56}{65}$ c. $-\frac{33}{65}$ d. $-\frac{16}{65}$ e. $-\frac{63}{16}$ f. $-\frac{33}{56}$

9. a. $\frac{m^2-n^2}{m^2+n^2}$ b. $\frac{2mn}{m^2+n^2}$, $\frac{2mn}{m^2-n^2}$ 11. b. $\frac{12}{5}$, $\frac{5}{12}$

2.4 Trigonometric Ratios of Angle

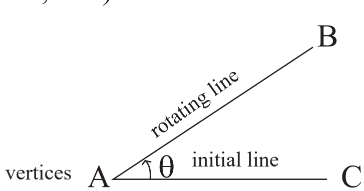
Do the following activity to develop the concept of positive and negative angles:

Activity 1

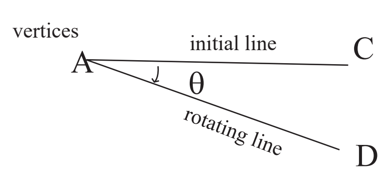
How can positive and negative angles be defined based on the direction? Explain with a diagram.

Using a ruler and a compass, draw a straight line AC. Call this line the initial line. From the left endpoint of the initial line, draw a rotated line at a certain angle ($\theta = 60°, 50°, 45°\ldots$, etc.) in the anti-clockwise direction.

In another diagram, from the initial line, draw a rotating line at a certain angle ($\theta = 60°, 50°, 45°\ldots$, etc.) in the clockwise direction.



anti-clockwise direction



clockwise direciton

In the figure $\angle CAB = \theta$ and $\angle CAD = (-\theta)$. Discuss what it might mean to write $\angle CAD = (-\theta)$

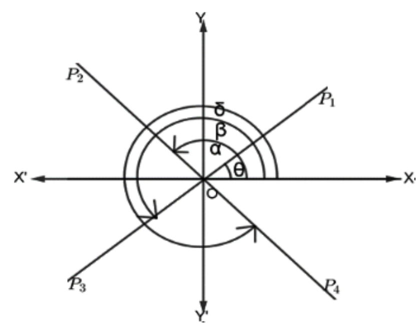
Here, the rotating line AB forms a positive angle θ with the initial line AC. Similarly, the rotating line AD forms a negative angle $(-\theta)$ with the initial line AC.

In the figure, $\angle CAB = \theta$ is formed in anti-clockwise direction, so, θ is positive. $\angle CAD = (-\theta)$ is formed in clockwise direction, so it is negative.

Therefore, the angle formed when the rotating line turns anti-clockwise direction from the initial line is called a positive angle.

Activity 2

Sit in suitable groups. Draw straight lines XX' and YY' intersect at point O such that the angle between them is 90°, and discuss the following questions:



a. In how many parts is the plane divided when they intersect in this way?

b. What is each part called?

c. What is the measure of the angle formed in each part?

d. If OX is taken as the initial line and the rotating line OP is rotated anti-clockwise (opposite to the direction of the clock's hands), in which quadrants are the angles θ, α, β, and δ formed respectively, and what could be the possible values of each? Discuss.

It can be seen that the two straight lines XOX' and YOY' divide the plane into four equal parts, and each part is called a quadrant. In each part, an angle of 90° is formed.

The upper right part from the X- axis, XOY is called the first quadrant. It includes angles from 0° to 90°. In the figure, $\angle XOY = 90°$.

The upper left part from the Y- axis, X'OY is called the second quadrant. It includes angles from 90° to 180°. In the figure, $\angle X'OY = 90°$.

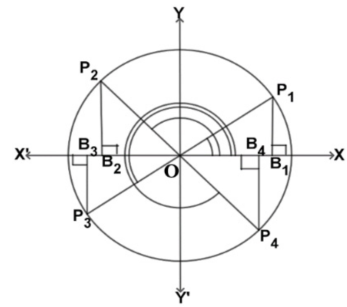
The lower left part from the X- axis, X'OY' is called the third quadrant. It includes angles from 180° to 270°. In the figure, $\angle X'OY' = 90°$.

The lower right part from the X- axis, XOY' is called the fourth quadrant. It includes angles from 270° to 360°. In the figure, $\angle XOY' = 90°$.

In this way, the angles θ, α, β, and δ are formed respectively in the first quadrant, second quadrant, third quadrant and fourth quadrant.

2.4.1 Trigonometric Ratios of Angles in All Quadrants

Activity 3



Sit in suitable groups. As shown in the figure, draw the lines XOX' and YOY' so that they intersect at point O at right angles. Take OX as the initial line and rotate the rotating line OP in the anti-clockwise direction. When rotated this way, make the angles XOP_1, XOP_2, XOP_3 and XOP_4 as shown in the figure. Those angles XOP_1, XOP_2, XOP_3 and XOP_4 lie respectively in the first, second, third and fourth quadrants. Why are these angles positive? In what situation would those same angles be negative? Discuss in groups.

All these angles are called positive angles because they are formed by rotating a line in the anti-clockwise direction. If the rotating line were rotated in the clockwise direction, the angles formed would be negative.

In the above figure, from the points labeled P_1, P_2, P_3 and P_4 on the rotating line, perpendiculars P_1B_1, P_2B_2, P_3B_3 and P_4B_4 are drawn to XOX' respectively.

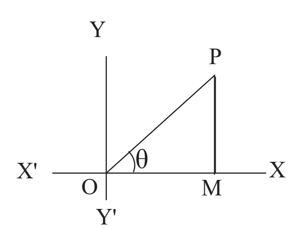
a. Since OP_1 lies in the first quadrant, both OB_1 and B_1P_1 are positive.
b. Since OP_2 lies in the second quadrant, OB_2 is negative because it lies to the left of the origin and B_2P_2 is positive because it is perpendicular above the X- axis.
c. Since OP_3 lies in the third quadrant, OB_3 is negative because it lies to the left of the origin and B_3P_3 is negative because it is perpendicular below the X- axis.
d. Since OP_4 lies in the fourth quadrant, OB_4 is positive because it lies to the right of the origin and B_4P_4 is negative because it is perpendicular below the X- axis.

Why do we always take OP_1, OP_2, OP_3 and OP_4 as positive? Discuss.

Now, how can we define the trigonometric ratios for angles that lie in different quadrants? In the figure, let a rotating line OP make an angle POM = θ in the anti-clockwise direction. From point P, draw perpendicular PM to OX. In this situation, trigonometric ratios can be defined as follows but the signs of the ratios vary according to the quadrant. Remember it.

$\sin\theta = \frac{PM}{OP}$ $\quad \cos\theta = \frac{OM}{OP}$ $\quad \tan\theta = \frac{PM}{OM}$

$\text{cosec}\theta = \frac{OP}{PM}$ $\quad \sec\theta = \frac{OP}{OM}$ $\quad \cot\theta = \frac{OM}{PM}$



We know that OP is always positive, but why? But PM and OM can be positive or negative depending on the quadrant, the values of the trigonometric ratios in different quadrants are as follows.

First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
The values of the all trigonometric ratios, are positive.	The values of the two trigonometric ratios, sin θ and cosec θ, are positive. The values of the remaining trigonometric ratios are negative.	The values of the two trigonometric ratios, tan θ and cot θ, are positive. The values of the remaining trigonometric ratios are negative.	The values of the two trigonometric ratios, cos θ and sec θ, are positive. The values of the remaining trigonometric ratios are negative.

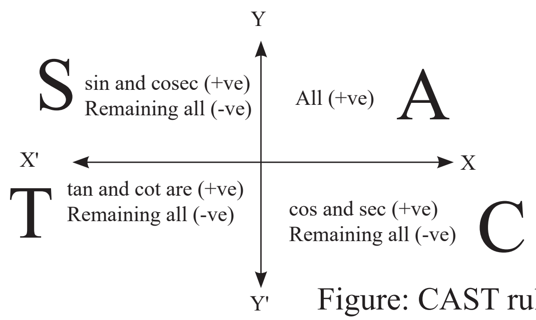
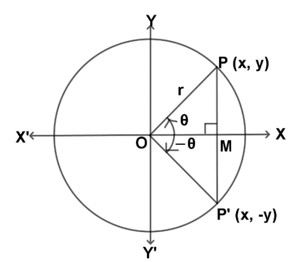


Figure: CAST rule

2.4.2 Trigonometric Ratios of Negative Angle (–θ)

Sit in an appropriate group. Draw the initial line OX as shown in the adjoining figure. Let the rotating line OP make an angle ∠POX = θ and take the coordinates of point P as P(x, y). Why are both the x- coordinate and y- coordinate of point P positive? Discuss.



Draw a circle with radius OP = r. From point P, draw a perpendicular PM to OX, and extend PM to meet the circle at point P', whose coordinates are P'(x, $-y$).

Why is the x- coordinate of P' positive and the y- coordinate negative? The line OP' makes an angle ∠P'OX = (– θ) with OX.

Similarly, why would the value of ∠P'OX be (– θ) instead of θ? Discuss.

According to definition. In right angled triangle PMO,

$$\sin\theta = \frac{\text{y- coordinate of P}}{\text{radius (OP)}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{x- coordinate of P}}{\text{radius (OP)}} = \frac{x}{r}$$

Thus, $$\sin(-\theta) = \frac{\text{y- coordinate of P'}}{\text{radius (OP')}} = \frac{-y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{\text{x- coordinate of P'}}{\text{radius (OP')}} = \frac{x}{r} = \cos\theta \qquad \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Similarly, how can the values of the other trigonometric ratios be determined for –θ? Discuss.

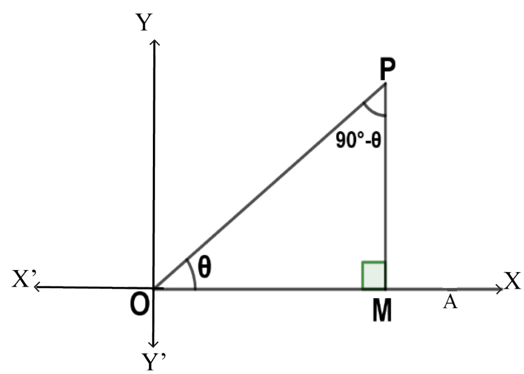
In short

The x- coordinate and y- coordinate of point P are both positive because both coordinates lie in the first quadrant. i.e. P(x, y)

sin(– θ) = – sinθ	cos(– θ) = cosθ	tan(– θ) = – tanθ
cosec(– θ) = – cosecθ	sec(– θ) = secθ	cot(– θ) = – cotθ

2.4.3 Trigonometric Ratio of (90° – θ)

Sit in an appropriate group. As shown in the adjoining figure, take the origin O as the center and draw the initial line OA. From the initial line, let the rotating line OP make an angle ∠POA = θ in the anticlockwise direction. From point P, draw a perpendicular PM to the initial line OA, forming triangle OPM.



We know that the sum of the three interior angles of a triangle is 180°. In the figure, since ∠PMO = 90°, the sum of the remaining two angles MOP and OPM is also 90°.

Now, ∠MPO = (90° – θ)

Taking ∠POA = θ as the reference angle, the hypotenuse is OP, the perpendicular is PM and the base is OM.

$\sin\theta = \frac{PM}{OP}$ $\cos\theta = \frac{OM}{OP}$ $\tan\theta = \frac{PM}{OM}$

$\text{cosec}\,\theta = \frac{OP}{PM}$ $\sec\theta = \frac{OP}{OM}$ $\cot\theta = \frac{OM}{PM}$

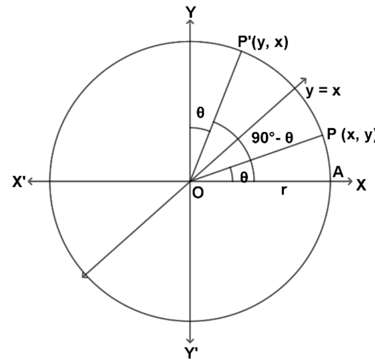
Similarly as, ∠MPO = (90 – θ) as the reference angle, the hypotenuse (h) is OP, the perpendicular (p) is OM and the base (b) is PM.

$\sin(90° - \theta) = \frac{p}{h} = \frac{OM}{OP} = \cos\theta$ $\cos(90° - \theta) = \frac{b}{h} = \frac{PM}{OP} = \sin\theta$

$\tan(90° - \theta) = \frac{p}{b} = \frac{OM}{PM} = \cot\theta$ $\text{cosec}(90° - \theta) = \frac{h}{p} = \frac{OP}{OM} = \sec\theta$

$\sec(90° - \theta) = \frac{h}{b} = \frac{OP}{PM} = \text{cosec}\,\theta$ $\cot(90° - \theta) = \frac{b}{p} = \frac{PM}{OM} = \tan\theta$

Alternative method



As in the adjoining figure, take the origin O as the center and draw the initial line OX. Let the rotating line OP make an angle ∠POX = θ. Draw a circle with radius OP = r.

Reflect the point P(x, y) over the line y = x. When such a transformation is applied, the reflected point P' has coordinates (y, x), and ∠P'OY = ∠POX = θ.

Therefore, in the figure, ∠P'OX = 90° – θ.

Now, from the definition of trigonometric ratios,

$$\sin\theta = \frac{\text{y- coordinate of P}}{\text{radius (OP)}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{x- coordinate of P}}{\text{radius (OP)}} = \frac{x}{r}$$

$$\text{Thus, } \sin(90°–\theta) = \sin\angle P'OX = \frac{\text{y- coordinate of P'}}{\text{radius (OP')}} = \frac{x}{OP} = \frac{x}{r} = \cos\theta$$

$$\text{Again, } \cos(90° – \theta) = \cos\angle P'OX = \frac{\text{x- coordinate of P'}}{\text{radius (OP')}} = \frac{y}{OP} = \frac{y}{r} = \sin\theta$$

$$\text{Again, } \tan(90° – \theta) = \frac{\sin (90° – \theta}{\cos (90° – \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$\cot(90° – \theta) = \frac{\cos (90° – \theta}{\sin (90° – \theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

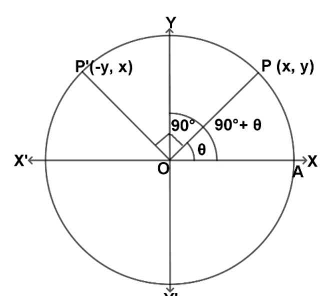
$$\text{Similarly, } \sec(90° – \theta) = \frac{1}{\cos (90° – \theta)} = \frac{1}{\sin\theta} = \text{cosec}\theta$$

$$\text{cosec}(90° – \theta) = \frac{1}{\sin (90° – \theta)} = \frac{1}{\cos\theta} = \sec\theta$$

In short

sin (90° – θ) = cos θ	cos (90° – θ) = sin θ	tan (90° – θ) = cot θ
cosec (90° – θ) = sec θ	sec (90° – θ) = cosec θ	cot (90°– θ) = tan θ

2.4.4 Trigonometric Ratio of (90° + θ)



As in the adjoining figure, take the origin O as the center and draw the initial line OX. Let the rotating line OP make an angle ∠POX = θ. Let the coordinate of P be (x, y). Draw a circle with radius OP = r.

When point P(x, y) is rotated 90° in the positive direction about the center O, the resulting image P' has coordinates (–y, x) and ∠P'OX = 90° + θ.

Now, from the definition of trigonometric ratios,

$$\sin\theta = \frac{\text{y- coordinate of P}}{\text{radius (OP)}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{x- coordinate of P}}{\text{radius (OP)}} = \frac{x}{r}$$

Thus, $\sin(90° + \theta) = \sin\angle P'OX = \frac{\text{y- coordinate of P'}}{\text{radius (OP')}} = \frac{x}{OP} = \frac{x}{r} = \cos\theta$

Again, $\cos(90° + \theta) = \cos\angle P'OX = \frac{\text{x- coordinate of P'}}{\text{radius (OP')}} = -\frac{y}{OP} = -\frac{y}{r} = -\sin\theta$

Similarly, $\tan(90° + \theta) = \frac{\sin(90° + \theta)}{\cos(90° + \theta)} = \frac{\cos\theta}{-\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$

$$\cot(90° + \theta) = \frac{1}{\tan(90° + \theta)} = \frac{1}{-\cot\theta} = -\tan\theta$$

$$\sec(90° + \theta) = \frac{1}{\cos(90° + \theta)} = \frac{1}{-\sin\theta} = -\mathrm{cosec}\theta$$

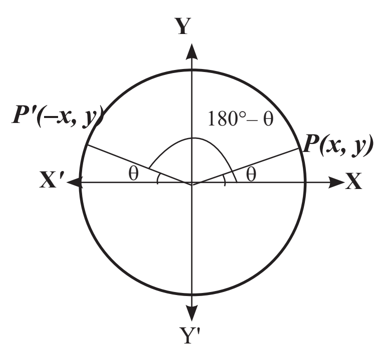
$$\mathrm{cosec}(90° + \theta) = \frac{1}{\sin(90° + \theta)} = \frac{1}{\cos\theta} = \sec\theta$$

In short

$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$	$\tan(90° + \theta) = -\cot\theta$
$\mathrm{cosec}(90° + \theta) = \sec\theta$	$\sec(90° + \theta) = -\mathrm{cosec}\theta$	$\cot(90° + \theta) = -\tan\theta$

2.4.5 Trigonometric Ratio of (180° – θ)

As in the adjoining figure, take the origin O as the center and draw the initial line OX. Let the rotating line OP make an angle $\angle POX = \theta$. Let the coordinate of P be (x, y). Draw a circle with radius $OP = r$.



Reflect the point $P(x, y)$ under Y- axis. When such a transformation is applied, the reflected point P' has coordinates $P'(-x, y)$, and $\angle P'OX' = \theta$.

Therefore, $\angle P'OX = 180° - \theta$.

$$\sin\theta = \frac{\text{y- coordinate of P}}{\text{radius (OP)}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{x- coordinate of P}}{\text{radius (OP)}} = \frac{x}{r}$$

Thus, $\sin(180° - \theta) = \frac{\text{y- coordinate of P'}}{\text{radius (OP')}} = \frac{y}{OP} = \frac{y}{r} = \sin\theta$

$$\cos(180° - \theta) = \frac{\text{x- coordinate of P'}}{\text{radius (OP')}} = -\frac{x}{OP} = -\frac{x}{r} = -\cos\theta$$

$$\text{Then, } \tan(180° - \theta) = \frac{\sin(180° - \theta)}{\cos(180° - \theta)} = \frac{\sin\theta}{-\cos\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$\cot(180° - \theta) = \frac{1}{\tan(180° - \theta)} = \frac{1}{-\tan\theta} = -\cot\theta$$

$$\text{Similarly, } \sec(180° - \theta) = \frac{1}{\cos(180° - \theta)} = \frac{1}{-\cos\theta} = -\sec\theta$$

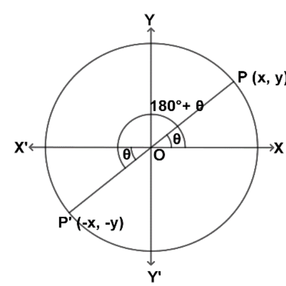
$$\operatorname{cosec}(180° - \theta) = \frac{1}{\sin(180° - \theta)} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

In short

sin (180° – θ) = sin θ	cos (180° – θ) = – cos θ	tan (180° – θ) = – tan θ
cosec (180°– θ) = cosec θ	sec (180° – θ) = – sec θ	cot (180° – θ) = – cot θ

2.4.6 Trigonometric Ratio of (180° + θ)

In the adjoining figure, take the origin O as the center and draw the initial line OX. Let the rotating line OP make an angle ∠POX = *θ*. Let the coordinate of P be (*x*, y). Draw a circle with radius OP = *r*.



When point P(*x*, y) is rotated 180° in the positive direction about the center O, the resulting image P' has coordinates P'(–*x*, –y) and ∠P'OX' = 180° + θ.

Now, from the definition of trigonometric ratios,

$$\sin\theta = \frac{\text{y- coordinate of P}}{\text{radius (OP)}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{x- coordinate of P}}{\text{radius (OP)}} = \frac{x}{r}$$

$$\text{Thus, } \sin(180° + \theta) = \frac{\text{y- coordinate of P'}}{\text{radius (OP')}} = \frac{-y}{OP} = \frac{-y}{r} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{\text{x- coordinate of P'}}{\text{radius (OP')}} = \frac{-x}{OP} = \frac{-x}{r} = -\cos\theta$$

$$\text{Again, } \tan(180° + \theta) = \frac{\sin(180° + \theta)}{\cos(180° + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\cot(180° + \theta) = \frac{1}{\tan(180° + \theta)} = \frac{1}{\tan\theta} = \cot\theta$$

$$\text{Similarly, } \sec(180° + \theta) = \frac{1}{\cos(180° + \theta)} = \frac{1}{-\cos\theta} = -\sec\theta$$

$$\operatorname{cosec}(180° + \theta) = \frac{1}{\sin(180° + \theta)} = \frac{1}{-\sin\theta} = -\operatorname{cosec}\theta$$

In short

$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$	$\tan(180° + \theta) = \tan\theta$
$\text{cosec}(180° + \theta) = -\text{cosec}\theta$	$\sec(180° + \theta) = -\sec\theta$	$\cot(180° + \theta) = \cot\theta$

Thus, we discussed what and how to determine the trigonometric ratios for a negative angle (–θ) and for (90° – θ), (90° + θ), (180° – θ), (180° + θ) also drew conclusions geometrically.

Now, similarly, discuss the remaining trigonometric ratios, such as (270° – θ), (270° + θ), (360° – θ), (360° + θ) and determine their values geometrically.

The formulas are presented below in a systematic manner.

2.4.7 Trigonometric Ratio of (270° – θ)

We know that, 270° = 180° + 90° Thus, (270° – θ) can be written as (180° + 90° – θ)

Now, $\sin(270° - \theta) = \sin\{180° + (90° - \theta)\} = -\sin(90° - \theta) = -\cos\theta$

$\cos(270° - \theta) = \cos\{180° + (90° - \theta)\} = -\cos(90° - \theta) = -\sin\theta$

Again, $\tan(270° - \theta) = \dfrac{\sin((270 \ \ - \theta)}{\cos(270 \ \ - \theta)} = \dfrac{-\cos\theta}{-\sin\theta} = \cot\theta$

$\cot(270° - \theta) = \dfrac{1}{\tan(270° - \theta)} = \dfrac{1}{\cot\theta} = \tan\theta$

Similarly, $\sec(270° - \theta) = \dfrac{1}{\cos(270° - \theta)} = \dfrac{1}{-\sin\theta} = -\text{cosec}\,\theta$

$\text{cosec}(270° - \theta) = \dfrac{1}{\sin(270° - \theta)} = \dfrac{1}{-\cos\theta} = -\sec\theta$

2.4.8 Trigonometric Ratios of (270° + θ)

Here, $\sin(270° + \theta) = \sin\{180° + (90° + \theta)\} = -\sin(90° + \theta) = -\cos\theta$

$\cos(270° + \theta) = \cos(180° + (90° + \theta) = -\cos(90° + \theta) = -(-\sin\theta) = \sin\theta$

$\tan(270° + \theta) = \dfrac{\sin((270° + \theta)}{\cos(270\square + \theta)} = \dfrac{-\cos\theta}{\sin\theta} = -\cot\theta$

$\cot(270° + \theta) = \dfrac{1}{\tan(270° + \theta)} = \dfrac{1}{-\cot\theta} = -\tan\theta$

Similarly, $\sec(270° + \theta) = \dfrac{1}{\cos(270° + \theta)} = \dfrac{1}{\sin\theta} = \text{cosec}\,\theta$

$\text{cosec}(270° + \theta) = \dfrac{1}{\sin(270° + \theta)} = \dfrac{1}{-\cos\theta} = -\sec\theta$

2.4.9 Trigonometric Ratios of (360° – θ)

Here, $\sin (360° - \theta) = \sin \{270° + (90° - \theta)\} = - \cos (90° - \theta) = -\sin \theta$

$\cos (360° - \theta) = \cos \{270° + (90° - \theta)\} = \sin (90° - \theta) = \cos \theta$

$$\cot (360° - \theta) = \frac{1}{\tan(360° - \theta)} = \frac{1}{-\tan\theta} = -\cot \theta$$

Similarly, $$\sec (360° - \theta) = \frac{1}{\cos(360° - \theta)} = \frac{1}{-\cos\theta} = \sec \theta$$

$$\text{cosec} (360° - \theta) = \frac{1}{\sin(360° - \theta)} = \frac{1}{-\sin\theta} = -\text{cosec } \theta$$

2.4.10 Trigonometric Ratio of (360° + θ)

We can take (360° + θ) as θ because trigonometric functions are periodic functions.

Therefore, $\sin (360° + \theta) = \sin \{270° + (90° + \theta)\} = - \cos (90° + \theta) = \sin \theta$

$\cos (360° + \theta) = \cos (270° + (90° + \theta) = \sin (90° + \theta) = \cos \theta$

$$\tan (360° + \theta) = \frac{\sin(360° + \theta)}{\cos(360° + \theta)} = \frac{\sin\theta}{\cos\theta} = \tan \theta$$

$$\cot (360° + \theta) = \frac{1}{\tan(360° + \theta)} = \frac{1}{\tan\theta} = \cot \theta$$

$$\sec (360° + \theta) = \frac{1}{\cos(360° + \theta)} = \frac{1}{\cos\theta} = \sec \theta$$

$$\text{cosec} (360° + \theta) = \frac{1}{\sin(360° + \theta)} = \frac{1}{\sin\theta} = \text{cosec}\theta$$

Generalized rule for trigonometric ratios of (n × 90° ± θ)

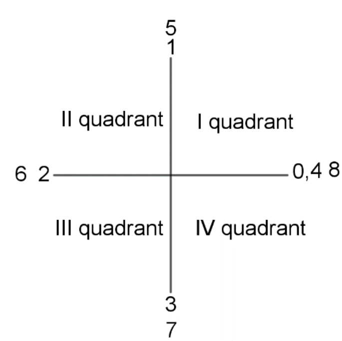
We have already established many relationships among trigonometric ratios, which can be generalized as shown below. Example:

a. i. $\sin (-\theta) = \sin (0 \times 90° - \theta) = - \sin \theta$
   ii. $\cos (-\theta) = \cos (0 \times 90° - \theta) = \cos\theta$
   iii. $\tan (-\theta) = \tan (0 \times 90° - \theta) = - \tan\theta$

b. i. $\sin (180° - \theta) = \sin (2 \times 90° - \theta) = \sin\theta$
   ii. $\cos (180° - \theta) = \cos (2 \times 90° - \theta) = - \cos\theta$
   iii. $\tan (180° - \theta) = \tan (2 \times 90° - \theta) = - \tan\theta$

c. i. $\sin (180° + \theta) = \sin (2 \times 90° + \theta) = -\sin\theta$
   ii. $\cos (180° + \theta) = \cos (2 \times 90° + \theta) = -\cos\theta$
   iii. $\tan (180° + \theta) = \tan (2 \times 90° + \theta) = \tan\theta$

d. i. $\sin (360° - \theta) = \sin (4 \times 90° - \theta) = - \sin\theta$
   ii. $\cos (360° - \theta) = \cos (4 \times 90° - \theta) = \cos\theta$



iii. $\tan (360° - \theta) = \tan (4 \times 90° - \theta) = -\tan\theta$

e. i. $\sin (360° + \theta) = \sin (4 \times 90° + \theta) = \sin\theta$

ii. $\cos (360° + \theta) = \cos (4 \times 90° + \theta) = \cos\theta$

iii. $\tan (360° + \theta) = \tan (4 \times 90° + \theta) = \tan\theta$

From the above result, the following rules can be written:

1. For multiples of 90°, when *n* is an even number (i.e., *n* = 2, 4, 6, …), the trigonometric ratios do not change, i.e., they remain the same.

   For example: sin → sin, cos → cos, tan → tan

   cosec → cosec, sec → sec, cot → cot
2. The signs (+, –) follow the CAST rule.
3. The trigonometric ratios of $(n \times 90° \pm \theta)$ are true for any even value of n. Then,

a. i. $\sin(90° - \theta) = \sin (1 \times 90° - \theta) = \cos \theta$

ii. $\cos(90° - \theta) = \cos (1 \times 90° - \theta) = \sin \theta$

iii. $\tan(90° - \theta) = \tan (1 \times 90° - \theta) = \cot \theta$

b. i. $\sin(90° + \theta) = \sin (1 \times 90° + \theta) = \cos \theta$

ii. $\cos(90° + \theta) = \cos (1 \times 90° + \theta) = -\sin \theta$

iii. $\tan(90° + \theta) = \tan (1 \times 90° + \theta) = -\cot \theta$

c. i. $\sin(270° - \theta) = \sin (3 \times 90° - \theta) = -\cos \theta$

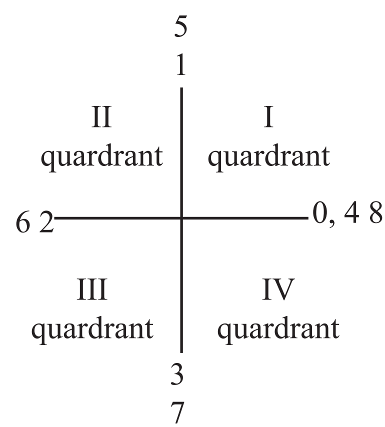
ii. $\cos(270° - \theta) = \cos (3 \times 90° - \theta) = -\sin \theta$

iii. $\tan(270° - \theta) = \tan (3 \times 90° - \theta) = \cot \theta$

d. i. $\sin(270° + \theta) = \sin (3 \times 90° + \theta) = -\cos \theta$

ii. $\cos(270° + \theta) = \cos (3 \times 90° + \theta) = \sin \theta$

iii. $\tan(270° + \theta) = \tan (3 \times 90° + \theta) = -\cot \theta$



From the above result, the following rules can be written:

1. For multiples of 90°, when *n* is an odd number (i.e., *n* = 1, 3, 5, …), the trigonometric ratios change, i.e., they do not remain the same.

   For example: sin → cos, cos → sin, tan → cot

   cosec → sec, sec → cosec, cot → tan

   Example: $\sin (270° - \theta) = \sin (3 \times 90° - \theta) = -\cos \theta$ lie on third quadrant, where, tanθ and cotθ are positive, but the remaining ratios are negative, so cosθ be negative.
2. The signs (+, –) follow the CAST rule.
3. The trigonometric ratios of $(n \times 90° \pm \theta)$ are true for any odd value of *n*.

Example 1

Prove that: $\tan\theta + \tan(180° - \theta) + \cot(90° + \theta) + \cot(90° - \theta) = 0$

Solution: Here,

L.H.S. $= \tan\theta + \tan(180° - \theta) + \cot(90° + \theta) + \cot(90° - \theta)$

$= \tan\theta + (-\tan\theta) + (-\tan\theta) + \tan\theta$

$= \tan\theta - \tan\theta - \tan\theta + \tan\theta$

$= 0$

= R.H.S. *proved.*

According to CAST, $(180 - \theta)$ and $(90 + \theta)$ lie on second quadrant, where sin and cosec only are positive, other remaining ratios are negative, so, $\tan(180 - \theta) = -\tan\theta$, $\cot(90+\theta) = -\tan\theta$.

Example 2

Prove that: $\sin^2\theta + \cos^2\alpha + \sin^2(90° - \theta) + \cos^2(90° - \alpha) = 2$

Solution: Here,

L.H.S. $= \sin^2\theta + \cos^2\alpha + \sin^2(90° - \theta) + \cos^2(90° - \alpha)$

$= \sin^2\theta + \cos^2\alpha + \{\sin(90° - \theta)\}^2 + \{\cos(90° - \alpha)\}^2$

$= \sin^2\theta + \cos^2\alpha + (\cos\theta)^2 + (\sin\theta)^2$

$= \sin^2\theta + \cos^2\theta + \cos^2\theta + \sin^2\theta$

$= 1 + 1 = 2$

= R.H.S. *proved.*

$\because$ According to CAST, $(90 - \theta)$ lies on first quadrant, where all trigonometric ratios are positive, so, $\sin(90 - \theta) = \cos\theta$, and $\cos(90 - \theta) = \sin\theta$.

Example 3

Find the value of $\tan 120°.\tan 135° + \sin 120°.\cos 180°$

Solution: Here,

$\tan 120°.\tan 135° + \sin 120°.\cos 180°$

$= \tan(180° - 60°)\tan(180° - 45°) + \sin(180° - 60°)\cos(180° - 0°)$

$= (-\tan 60°).(-\tan 45°) + \sin 60°(-\cos 0°)$

$= (-\sqrt{3}).(-1) + \frac{\sqrt{3}}{2}.(-1)$

$= \sqrt{3} - \frac{\sqrt{3}}{2}$

$= \frac{2\sqrt{3} - \sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\because (180 - \theta)$ lies on second quadrant, where sin and cosec only are positive, other remaining ratios are negative, so, $\sin(180 - \theta) = \sin\theta$, $\cos(180 - \theta) = -\cos\theta$ and $\tan(180 - \theta) = -\tan\theta$.

Example 4

Simplify: $\frac{\cos(270° - A).\sec(180° - A).\sin(270° + A)}{\cos(90° + A).\cos(180° - A).\sin(180° + A)}$

Solution: Here,

$= \dfrac{\cos(270° - A).\ \sec(180° - A).\ \sin(270° + A)}{\cos(90° + A).\ \cos(180° - A).\ \sin(180° + A)}$

$= \dfrac{(-\sin A).\ (-\sec A).\ (-\cos A)}{(-\sin A).\ (-\cos A).(-\sin A)}$ [According to CAST]

$= \dfrac{\sec A}{\sin A}$

$= \dfrac{1}{\sin A.\cos A}$

$= \text{cosec}A.\sec A$

Example 5

Prove that: $\sin 112° + \cos 74° - \sin 68° + \cos 106° = 0$

How can we get 0 on the RHS?
To make the RHS equal to 0, all the terms on the LHS must simplify in such a way that the positive and negative terms cancelling in each other

Solution: Here,

L.H.S.

$= \sin 112° + \cos 74° - \sin 68° + \cos 106°$

$= \sin(180° - 68°) + \cos 74° - \sin 68° + \cos(180° - 74°)$

$= \sin 68° + \cos 74° - \sin 68° + (-\cos 74°)$

$= \sin 68° + \cos 74° - \sin 68° - \cos 74°$

$= 0$

= R.H.S. *proved.*

$\because$ (180 – θ) lies on second quadrant, where sin and cosec only are positive, other remaining ratios are negative, so, $\sin(180 - \theta) = \sin\theta$, $\cos(180 - \theta) = -\cos\theta$.

Example 6

Find the value of $2\cos^2 135° + \sin 150° + \dfrac{1}{2}\sin 180° + \tan^2 135°$

Solution: Here,

$= 2\cos^2 135° + \sin 150° + \dfrac{1}{2}\sin 180° + \tan^2 135°$

$= 2\{\cos(180° - 45°)\}^2 + \sin(180° - 30°) + \dfrac{1}{2}\sin(180° - 0°) + \{\tan(180° - 45°)\}^2$

$= 2(-\cos 45°)^2 + \sin 30° + \dfrac{1}{2}\sin 0° + (-\tan 45°)^2$ {$\because$ According to CAST}

$= 2\left(-\dfrac{1}{\sqrt{2}}\right)^2 + \dfrac{1}{2} + \dfrac{1}{2} \times 0 + (-1)^2$

$= 2 \times \dfrac{1}{2} + \dfrac{1}{2} + 0 + 1$

$= 2 + \dfrac{1}{2} = \dfrac{5}{2} = 2\dfrac{1}{2}$

Example 7

Prove that: $\cos 240° \sin 300° - \sin 330° \cos 300° = \dfrac{\sqrt{3} + 1}{4}$

Solution: Here,

L.H.S. $= \cos 240° \sin 300° - \sin 330° \cos 300°$

$= \cos(180° + 60°) \sin(360° - 60°) - \sin(360° - 30°) \cos(360° - 60°)$

$= (-\cos 60°)(-\sin 60°) - (-\sin 30°) \cos 60°$ {∵ According to CAST}

$= \left(\frac{-1}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) \square \left(\frac{-1}{2}\right)\frac{1}{2}$

$= \frac{\sqrt{3}}{4} + \frac{1}{4}$

$= \frac{\sqrt{3}+1}{4}$ = R.H.S. *Proved*

Example 8

Prove that: $\cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \cos\frac{5\pi^c}{8} + \cos\frac{7\pi^c}{8} = 0$

Solution: Here,

L.H.S $= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \cos\frac{5\pi^c}{8} + \cos\frac{7\pi^c}{8}$ {∵ Arranging the terms}

$= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \cos\frac{8\pi^c - 3\pi^c}{8} + \cos\frac{8\pi^c - \pi^c}{8}$

$= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \cos\left(\frac{8\pi^c}{8} - \frac{3\pi^c}{8}\right) + \cos\left(\frac{8\pi^c}{8} - \frac{\pi^c}{8}\right)$

$= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \cos\left(\pi^c - \frac{3\pi^c}{8}\right) + \cos\left(\pi^c - \frac{\pi^c}{8}\right)$

$= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} + \left\{-\cos\left(\frac{3\pi^c}{8}\right)\right\} + \left\{-\cos\left(\frac{\pi^c}{8}\right)\right\}$ [According to CAST, $\cos(180 - \theta) = -\cos\theta$]

$= \cos\frac{\pi^c}{8} + \cos\frac{3\pi^c}{8} - \cos\frac{3\pi^c}{8} - \cos\frac{\pi^c}{8}$

$= 0$

= R.H.S. *proved.*

Example 9

Find the value of x in the given condition :

$x \cot A. \tan(90° + A) = \tan(90° + A). \cot(180° - A) + x \sec(90° + A) \operatorname{cosec} A$

Solution: Here,

Or, $x \cot A. (-\cot A) = (-\cot A)(-\cot A) + x(-\operatorname{cosec} A) \operatorname{cosec} A$

Or, $-x \cot^2 A = \cot^2 A - x \operatorname{cosec}^2 A$

Or, $x \operatorname{cosec}^2 A - x \cot^2 A = \cot^2 A$

Or, $x(\operatorname{cosec}^2 A - \cot^2 A) = \cot^2 A$

Or, $x \times 1 = \cot^2 A$

Or, $x = \cot^2 A$

$\therefore x = \cot^2 A$

Exercise 2.4

1. Fill in the blanks.

   a. $\sin(-\theta) = \cdots\cdot$ b. $\tan(-\theta) = \cdots\cdot$ c. $\sin(90° - A) = \cdots\cdot$

   d. $\sin(90° + A) = \cdots\cdot$ e. $\sec(180° - \theta) = \cdots\cdot$ f. $\tan(180° + \theta) = \cdots\cdot$

   g. $\cos(270° - \theta) = \cdots\cdot$ h. $\cot(270° + \theta) = \cdots\cdot$ i. $\text{cosec}(360° - \theta) = \cdots$

2. Match the following.

Sin (90° + A)	– cosA
cos (180° – A)	– cot A
tan (180° + A)	cosA
cosec (270° – A)	tan A
Sec (270° + A)	cosecA
Cot (360° – A)	– secA
Cot (360° + A)	cotA

3. Explain the concept of CAST with the help of a diagram.

4. Calculate the following.

   a. sec 150° b. tan 120° c. sin 225° d. sec 210° e. tan 240°

   f. cot 210° g. cos 870° h. sin 1230° i. cosec (–1200°)

5. Prove that:

   a. $\sec A . \text{cosec}(90° - A) - \tan A \cot(90° - A) = 1$

   b. $\sin\theta. \sec\left(\frac{\pi^c}{2} - \theta\right) - \text{cosec}\theta. \cos\left(\frac{\pi^c}{2} - \theta\right) = 0$ $\{\frac{\pi}{2} = 90°\}$

   c. $\tan\theta + \tan(180° - \theta) + \cot\left(\frac{\pi^c}{2} + \theta\right) - \cot\theta\left(\frac{\pi^c}{2} + \theta\right) = 0$

   d. $\sin\theta . \sin(180° - \theta) - \cos\theta . \cos(\pi^c - \theta) = 1$

6. Prove that:

   a. $\sin^2(\frac{\pi}{2}\ \square\theta). \text{cosec}\ \theta - \tan^2\left(\frac{\pi}{2} - \theta\right). \sin\theta = 0$

   b. $\tan^2\alpha . \sec^2(90° - \alpha) - cosec^2(90° - \alpha). \sin^2\alpha = 1$

   c. $\frac{\tan^2\theta}{\cos^2(90°-\theta)} - \frac{\sin^2\theta}{\sin^2(90°-\theta)} = 1$

7. Simplify:

a. $\dfrac{\sin(90° + \theta) \times \text{cosec}(90° + \theta)}{\cos\theta \times \cot(90° + \theta)}$

b. $\dfrac{\tan(90° + \theta) \times \sec(270° - \theta) \times \sin(-\theta)}{\cos(180°+\theta) \times \cos(-\theta)}$

c. $\dfrac{\sin(180° - \theta) . \tan(90° + \theta) . \sec(90° + \theta)}{\sin(90° + \theta) . \cos(180° - \theta) . \cot(180° - \theta)}$

d. $\dfrac{\sin(90° + \theta) . \cos(-\theta) . \cot(180° - \theta)}{\cos(360° - \theta) . \cos(180° + \theta) . \tan(90° - \theta)}$

e. $\dfrac{\tan(180° - \alpha) \times \cot(90° - \alpha) \times \cos(360° - \alpha)}{\sin(-\alpha) \times \tan(180° - \alpha) \times \tan(90° + \alpha)}$

f. $\dfrac{\cos(270° - \theta) \times \sec(180° - \theta) \times \sin(270° + \theta)}{\cos(90° + \theta) \times \cos(180° - \theta) \times \sin(180° + \theta)}$

8. Prove that:

a. $\cos 70°.\sin 20° + \cos 20°.\sin 70° = 1$

b. $\cos 72° + \cos 144° - \sin 18° + \sin 54° = 0$

c. $\sin 112° + \cos 74° + \cos 106° - \sin 68° = 0$

d. $\cos 20° + \cos 40° + \cos 140° + \cos 160° = 0$

e. $\cos 306° + \cos 234° + \cos 162° + \cos 18° = 0$

f. $\cos 12° + \cos 64° + \cos 116° + \cos 168° = 0$

9. Find the value of:

a. $2\cos^2 45° + \sin 30° + \frac{1}{2}\cos 180° - \tan 45°$

b. $\cos^2 120° + \sin^2 150° + \sin 120° + \cos 150°$

c. $\sin^2 120° - \sin^2 135° - \tan^2 150° - \cos^2 120°$

d. $\sin^2 180° + \sin^2 150° + \sin^2 135° + \sin^2 120° + \sin^2 90°$

e. $\sec^2 180° + 2\cos^2 45° + \text{cosec}^2 135° + \tan^2 45° - 4\sin^2 60°$

10. Prove that:

a. $\cos 120° \times \text{Sin } 150° + \cos 330° \times \sin 300° = -1$

b. $\sin 420° \times \cos 390° + \cos(-300°) \times \sin(-330°) = 1$

c. $\cos 240° \times \sin 300° - \sin 330° \times \cos 300° = \dfrac{1+\sqrt{3}}{4}$

d. $\cos 570° . \sin 510° + \sin 330° . \cos 390° = \dfrac{-\sqrt{3}}{2}$

e. $\sin 660° \times \sin 330° + \cos 120° \times \sin 150° = \dfrac{\sqrt{3}-1}{4}$

11. Prove that:

a. $\sin^2\frac{\pi^c}{4} + \sin^2\frac{3\pi^c}{4} + \sin^2\frac{5\pi^c}{4} + \sin^2\frac{7\pi^c}{4} = 2$

b. $\cos^2\frac{\pi^c}{8} + \cos^2\frac{3\pi^c}{8} + \cos^2\frac{5\pi^c}{8} + \cos^2\frac{7\pi^c}{8} = 2$

c. $\sin^2\frac{\pi^c}{8} + \sin^2\frac{3\pi^c}{8} + \sin^2\frac{5\pi^c}{8} + \sin^2\frac{7\pi^c}{8} = 2$

12. Find the value of x in the given condition.

a. $x \tan 150° + x \sin 120°.\cos 180° = 2 \cot 120°$

b. $x \cos A.\ \cot (90° + A) - \sin (90° + A) + \text{cosec} (90° + A) = 0$

c. $x \tan \theta\ .\ \cot (90° + \theta) - \cot (90° + \theta).\ \tan (180° - \theta) = \sec \theta\ \text{cosec} (270° - \theta)$

d. $\sec (90° + A)\ \text{cosec}\ A + \tan (90° + A) \cot (180° - A) = x \cot A \tan (90° + A)$

e. $x \tan (180° + \theta).\ \cot (90° - \theta) = \tan (180° - \theta).\ \tan (360° - \theta) + x\ \text{cosec} (90° - \theta).\ \text{cosec} (90° + \theta)$

Project Work

Problem: Based on the first, second, third, and fourth quadrants, study and write the relationship of trigonometric ratios with their respective positive (+) and negative (–) signs. For this, explore the trigonometric identities involving (90° ± θ), (180° ± θ), (270° ± θ), (360° ± θ) and (–θ) so that identifying the signs of trigonometric ratios in each quadrant becomes easier.

Required materials: A4-size paper, chart paper, graph paper, and coloured pens or markers, etc.

Process: Include the subject matter facilitated by the teacher in the classroom, and have students write the CAST rule with the help of various textbooks and the internet.

Write the relationship of trigonometric ratios with (90° ± θ), (180° ± θ), (270° ± θ), (360° ± θ) and (–θ). After study and discussion, present the prepared materials using chart paper, PowerPoint, or other appropriate methods. If suitable suggestions are received from the teacher and classmates, include those suggestions.

Time limit: As instructed by the teacher.

Answer

1. a. $-\sin\theta$ b. $-\tan\theta$ c. $\cos A$ d. $\cos A$ e. $-\sec\theta$ f. $\tan\theta$ g. $-\sin\theta$ h. $-\tan\theta$ i. $-\text{cosec}\theta$

2 - 3 Show to the teacher.

4. a. $-\frac{2}{\sqrt{3}}$ b. $-\sqrt{3}$ c. $-\frac{1}{\sqrt{2}}$ d. $-\frac{2}{\sqrt{3}}$ e. $\sqrt{3}$ f. $\sqrt{3}$ g. $-\frac{\sqrt{3}}{2}$ h. $\frac{1}{2}$ i. $-\frac{2}{\sqrt{3}}$

7. a. $-\text{cosec}\theta$ b. $\text{cosec}\theta\sec\theta$ c. $\sec^2\theta$ d. 1 e. $\tan\alpha$ f. $\sec\theta\ \text{cosec}\theta$

9. a. 0 b. $-\frac{1}{2}$ c. $-\frac{1}{3}$ d. $\frac{5}{2}$ e. 2

12. a. $\frac{4}{5}$ b. $\tan A$ c. $\cot^2\theta$ d. $\tan^2 A$ e. $-\tan^2\theta$

Lesson 3

Geometry

The French mathematician René Descartes connected algebra and geometry to invent a new branch of mathematics called coordinate geometry. The Cartesian coordinate system is named after him.

Every point on a plane surface is represented by numerical coordinates (x, y) in the Cartesian coordinate system. The development of coordinate geometry helped Sir Isaac Newton and Wilhelm Leibniz in the invention of calculus. The use of the Cartesian coordinate system is found in physics, computer science, engineering, and astronomy by providing a common language for algebra and geometry.

3.1 Concept of Locus

Activity 1

Problem:

x	0	1	…	…	…
$y = x + 1$	1	2	…	…	…

Complete the given table and plot the points (0, 1), (1, 2), … on the graph. Now, connect those points. What kind of figure is formed? Write the conclusion.

Required Materials: Graph paper

Process: Each student should connect the plotted points on the graph and observe what kind of line is formed. Discuss whether the same type of line is obtained or not, and confirm through discussion.

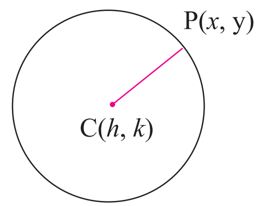
Conclusion: ....

When the above points are plotted and connected on the graph, a straight line is formed, which represents a locus. A locus is a set of points that satisfy a given condition or rule. In the above example, $y = x + 1$ is a locus. Therefore, a line, condition, or equation is used to represent a locus.

Let's look at an example: In the figure, the locus of a point P that lies at an equal distance from a fixed point C is a circle. The fixed point is the center of the circle, and the equal distance is its radius.

Here, if the moving point is $P(x, y)$, then the fixed condition is the equal distance.

Therefore, the locus of the point $P(x, y)$ is a circle.



If the center of the given circle is $C(h, k)$ and a point on the circumference is $P(x, y)$,

then,

by the distance formula, $r = \sqrt{(x-h)^2 + (y-k)^2}$

Squaring on both sides,

$r^2 = (x-h)^2 + (y-k)^2$ gives the equation of the circle, which is called the equation of the locus.

Example 1

If the point (1, 2) lies on the locus represented by the equation $x^2 + y^2 + kx + 3y + 6 = 0$, find the value of k.

Solution

Here, since $(x, y) = (1, 2)$ lies on the equation $x^2 + y^2 + kx + 3y + 6 = 0$

Or, $1^2 + 2^2 + k \times 1 + 3 \times 2 + 6 = 0$ [$\because$ substituting the value of x and y.]

Or, $1 + 4 + k + 6 + 6 = 0$

Or, $17 + k = 0$

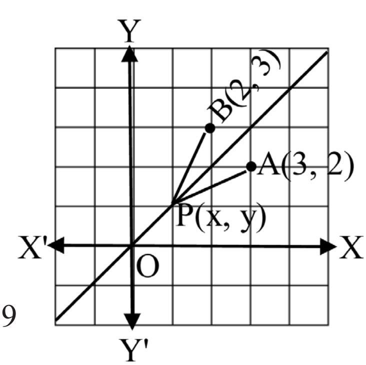
Or, $k = -17$

$\therefore$ $k = -17$

Example 2

Find the equation of the locus of a point such that its distances from (3, 2) and (2, 3) are equal.

Solution: Here,



Let the coordinates of the moving point P, which is equidistant from the points A(3, 2) and B(2, 3), be (x, y).

Now, AP = BP

Or, $AP^2 = BP^2$

Or, $(x-3)^2 + (y-2)^2 = (x-2)^2 + (y-3)^2$

Or, $x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 - 4x + 4 + y^2 - 6y + 9$

Or, $-6x + 4x - 4y + 6y = 0$

Or, $-2x + 2y = 0$

Or, $-2(x - y) = 0$

$\therefore$ $x - y = 0$ is the required equation of the locus.

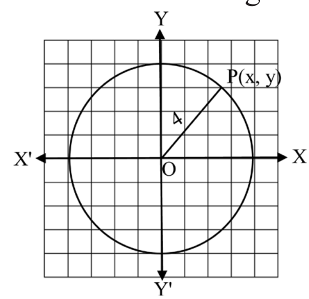
Thought Provoking Question: Where did the –2 disappear?

Example 3

Find the equation of the locus of a point that moves at a distance of 4 units from the origin.

Solution: Here,

Let, P(x, y) be a point that moves such that it is at a distance of 4 units from the origin O(0, 0).



Now, OP = 4

Or, $OP^2 = 4^2$

Or, $(x - 0)^2 + (y - 0)^2 = 16$

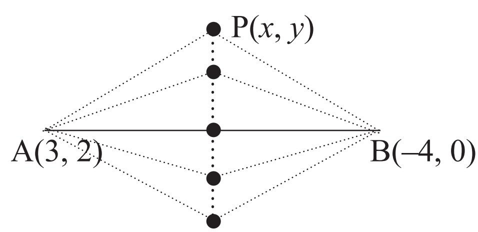
$\therefore$ $x^2 + y^2 = 16$ is the required equation of the locus.

Example 4

Find the equation of the locus of the perpendicular bisector of the line segment joining the points A(3, 2) and B(–4, 0).

Solution: Here,

Let the coordinates of the moving point P, which lies on the perpendicular bisector of the line segment AB joining A(3, 2) and B(–4, 0), be P(x, y).



Now, AP = BP

Or, $AP^2 = BP^2$

Or, $(x - 3)^2 + (y - 2)^2 = (x + 4)^2 + (y - 0)^2$

Or, $x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 + 8x + 16 + y^2$

Or, $-6x - 8x - 4y + 13 - 16 = 0$

Or, $-14x - 4y - 3 = 0$

$\therefore$ $14x + 4y - 3 = 0$ is the required equation of the locus.

Example 5

Find the equation of the locus of a moving point whose distance from the X- axis is twice its distance from the point (2, 4).

Solution: Here,

Let, A(x, 0) be a point on the X- axis, B(2, 4) be a fixed point and P(x, y) be the moving point.

According to question,

AP = 2BP

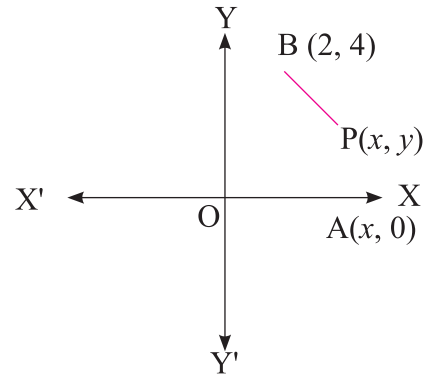
Or, $AP^2 = 4BP^2$

Or, $(x - x)^2 + (y - 0)^2 = 4\{(x - 2)^2 + (y - 4)^2\}$

Or, $y^2 = 4(x^2 - 4x + 4 + y^2 - 8y + 16)$

Or, $y^2 = 4x^2 - 16x + 4y^2 - 32y + 80$

Or, $4x^2 + 4y^2 - y^2 - 16x - 32y + 80 = 0$



$\therefore$ $4x^2 + 3y^2 - 16x - 32y + 80 = 0$ is the required equation of the locus.

Example 6

If A(2, 3) and B(–4, 7) are given points and P(x, y) is a moving point such that $PA^2 = AB^2$, find the equation of the locus.

Solution

Here, the given points are A(2, 3) and B(–4, 7). Let the coordinates of the moving point be P(x, y).

Now, $PA^2 = AB^2$

Or, $(x - 2)^2 + (y - 3)^2 = (-4 - 2)^2 + (7 - 3)^2$

Or, $x^2 - 4x + 4 + y^2 - 6y + 9 = 36 + 16$

Or, $x^2 + y^2 - 4x - 6y + 13 - 36 - 16 = 0$

$\therefore$ $x^2 + y^2 - 4x - 6y - 39 = 0$ is the required equation of the locus.

Exercise 3.1 (A)

1. What is locus? Write it.
2. Among the points (3, 2), (4, 3), (5, 0) and (0, –5), which points lie on the locus $x^2 + y^2 = 25$? Find it.
3. If the point (2, –1) lies on the locus of $kx^2 - 2y^2 - 2x + 3y - 3 = 0$, find the value of k.
4. If the point $(k - 1, k + 3)$ lies on the locus of $3x - y + 7 = 0$, find the value of k.
5. Find the equation of the locus of a point which is equidistant from the points (–2, 1) and (4, 1).
6. Find the equation of the locus of a point which is equidistant from the points (1, 2) and Y- axis.
7. Find the equation of the locus of a point that moves at a distance of 4 units from the origin.

8. Find the equation of the locus of a point which is equidistant from the points (–2, 5) and Y- axis.

9. Two fixed points A(7, 0) and B(−7, 0) are given, and P is a moving point. Find the equation of the locus of P such that $PA^2 + PB^2 = AB^2$.

10. Two points A(3, 2) and B(7, −4) are given. If P(x, y) is a moving point, find the equation of the locus in the following cases:

    a. PA = PB b. AP = 2PB c. $PA^2 = AB^2$

11. Find the equation of the locus of a point which moves such that its distance from the point (0, 2) is half of the distance from the point (2, –3).

12. Find the equation of the locus of a point which moves such that its distance from the point (1, 0) is double of the distance from the point (0, –2).

13. Find the equation of the locus of a point which moves such that its distance from the point (3, 0) is three times the distance from the point (0, 2).

14. Find the equation of the locus of a point which moves such that its distance from the point (3, 4) is double of the distance from the X- axis.

15. Find the equation of the locus of a point which moves such that its distance from the point (−2, 5) is half of the distance from the Y- axis.

Answer

1 - 2. Show to the teacher. 3. 3 4. $-\frac{1}{2}$ 5. $x = 1$

6. $y^2 - 2x - 4y + 5 = 0$ 7. $x^2 + y^2 - 16 = 0$ 8. $y^2 + 4x - 10y + 29 = 0$

9. $x^2 + y^2 = 49$ 10. a. $2x - 3y - 13 = 0$ b. $3x^2 + 3y^2 - 50x + 30y + 117 = 0$

c. $x^2 + y^2 - 6x - 4y - 39 = 0$ 11. $3x^2 + 3y^2 + 4x - 22y + 3 = 0$

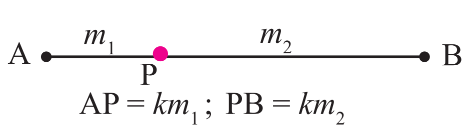
12. $3x^2 + 3y^2 + 2x + 16y + 15 = 0$ 13. $8x^2 + 8y^2 + 6x - 36y + 27 = 0$

14. $x^2 - 3y^2 - 6x - 8y + 25 = 0$ 15. $3x^2 + 4y^2 + 16x - 40y + 116 = 0$

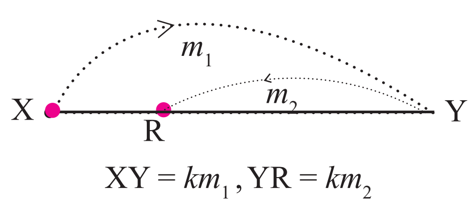
3.1.2 Point which Divides a Line Segment in a Ratio

a. PA = 2 cm, PB = 3 cm, AB = 5 cm b. RX = 10 cm, RY = 15 cm, XY = 25 cm

Suppose, AP = $k\,m_1$ and PB = $k\,m_2$, then the ratio between AP and PB is AP:PB = m_1:m_2.



Dividing AB internally by P.



Dividing XR externally by Y.

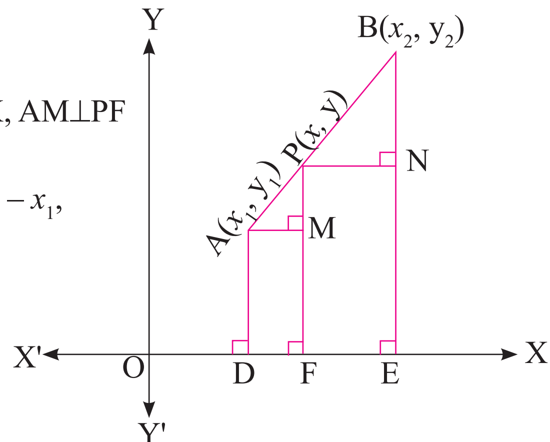
Similarly, XY = km_1 and RY = km_2, then the ratio between XY and RY is AP:PB = m_1:m_2. Point P divides AB internally because P lies on the interior of AB. Similarly, Y divides XR externally because Y lies on the exterior of XR.

Section Formula

Let, in the figure, $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points. $P(x, y)$ divides the line segment AB internally in the ratio m_1:m_2.

Calculate the coordinate of P(x, y).

Now, draw AD⊥OX, BE⊥OX, PF⊥OX, AM⊥PF and PN⊥BE



In the figure, AM = DF = OF – OD = $x - x_1$,

PN = FE = OE – OF = $x_2 - x$

PM = PF – MF = PF – AD = $y - y_1$

BN = BE – NE = BE – PF = $y_2 - y$

And $\frac{AP}{BP} = \frac{m_1}{m_2}$

Right angled triangles ΔPMA and ΔBNP are similar.

$\frac{AM}{PN} = \frac{AP}{BP}$ [∵ The ratio of corresponding sides of similar triangle]

Or, $\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}$

Or, $m_2x - m_2x_1 = m_1x_2 - m_1x$ Or, $m_2x + m_1x = m_1x_2 + m_2x_1$

Or, $x(m_1 + m_2) = m_1x_2 + m_2x_1$ Or, $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

Again, $\frac{PM}{BN} = \frac{AP}{BP}$ [∵ The ratio of corresponding sides of similar triangle]

Or, $\frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}$

Or, $m_2y - m_2y_1 = m_1y_2 - m_1y$

Or, $m_2y + m_1y = m_1y_2 + m_2y_1$

Or, $y(m_1 + m_2) = m_1y_2 + m_2y_1$

Or, $y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$

$\therefore (x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

Thus, the coordinates of the point dividing line segment AB are

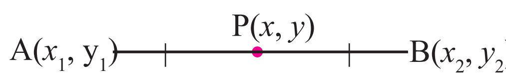
$P(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

Some Special Cases

Thought Provoking Question: If $m_1 = m_2$ in the above case, what kind of point P is?

A. If P (x, y) be the midpoint of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then AP = BP and AP:BP = $m_1:m_2 = 1:1$.

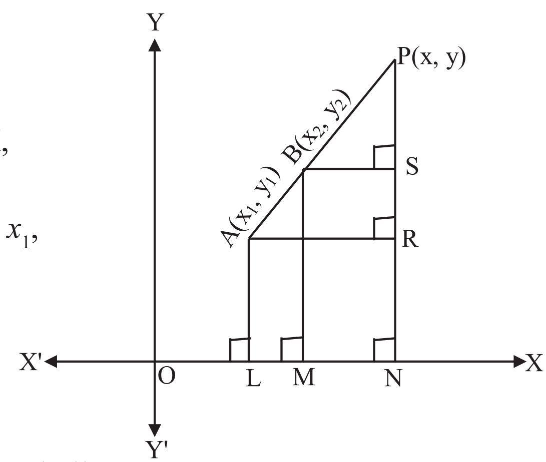
$$P(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$
$$= \left(\frac{1.x_2 + 1.x_1}{1+1}, \frac{1.y_2 + 1.y_1}{1+1}\right)$$
$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



∴ $P(x, y)$ is called the coordinate of midpoint.

B. If $P(x, y)$ divides the line segment joining points $A(x_1, y_1)$ and $B(x_2, z_2)$ externally in the ratio $m_1:m_2$ then AP:BP = $m_1:m_2$

Suppose, in the figure, the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is divided externally by $P(x, y)$ in the ratio $m_1:m_2$.



Or, $\frac{AP}{BP} = \frac{m_1}{m_2}$

Now, draw AL⊥OX, BM⊥OX, PN⊥OX, AR⊥PN and BS⊥PN

In the figure, AR = LN = ON – OL = $x - x_1$,

BS = MN = ON – OM = $x - x_2$

PR = PN – RN = PN – AL = $y - y_1$

PS = PN – SN = PN – BM = $y - y_2$

And $\frac{AP}{PB} = \frac{m_1}{m_2}$

Right angled triangles Δ PAR and Δ BSP are similar.

$\frac{AR}{BS} = \frac{AP}{BP}$ [∵ The ratio of corresponding sides of similar triangl]

Or, $\frac{x - x_1}{x - x_2} = \frac{m_1}{m_2}$

Or, $m_2(x - x_1) = m_1(x - x_2)$

Or, $m_2x - m_2x_1 = m_1x - m_1x_2$

Or, $m_1x_2 - m_2x_1 = m_1x - m_2x$

Or, $m_1x_2 - m_2x_1 = x(m_1 - m_2)$

Or, $x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$

Again, $\frac{PR}{PS} = \frac{AP}{BP}$ [$\because$ The ratio of corresponding sides of similar triangle]

Or, $\frac{y - y_1}{y - y_2} = \frac{m_1}{m_2}$

Or, $m_2 (y - y_1) = m_1(y - y_2)$

Or, $m_2y - m_2y_1 = m_1y - m_1y_2$

Or, $m_1y_2 - m_2y_1 = m_1y - m_2y$

Or, $m_1y_2 - m_2y_1 = y(m_1 - m_2)$

Or, $y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$

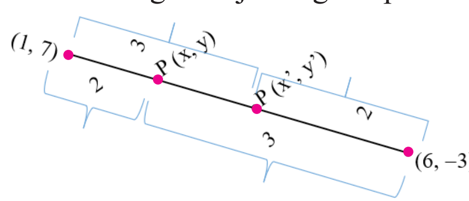
$\therefore (x, y) = (\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2})$

Thus, the coordinates of the point dividing line segment AB externally are $P(x, y)$

$= (\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2})$

Example 1

Find the coordinates of the point that divides the line segment joining the points (1, 7) and (6, –3) in the ratio 2:3.



Solution: Here,

Given points are, $(x_1, y_1) = (1, 7)$

$(x_2, y_2) = (6, -3)$

$m_1 : m_2 = 2:3$

$(x, y) = ?$

Now, according to formula,

$(x, y) = (\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2})$

$= (\frac{2\times6+3\times1}{2+3}, \frac{2\times(-3)+3\times7}{2+3})$

$= (\frac{12+3}{5}, \frac{-6+21}{5})$

$= (\frac{15}{5}, \frac{15}{5})$

$= (3, 3)$

Alternatively

If, $(x_1, y_1) = (6, -3)$ and

$(x_2, y_2) = (1, 7)$

$m_1 : m_2 = 2:3$

$(x, y) = ?$

Now by formula,

$(x, y) = (\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2})$

$= (\frac{2\times1+3\times6}{2+3}, \frac{2\times7+3\times(-3)}{2+3})$

$= (\frac{2+18}{5}, \frac{14-9}{5}) = (\frac{20}{5}, \frac{5}{5}) = (4, 1)$

Thus, the coordinates of the point that divides the line segment joining the points (1, 7) and (6, –3) in the ratio 2:3 are (3, 3) and (4, 1).

Example 2

Find the coordinates of the point that divides the line segment joining the points (5, –2) and (9, 6) externally in the ratio 3:1.

Solution: Here,

$(x_1, y_1) = (5, -2)$
$(x_2, y_2) = (9, 6)$
$m_1 : m_2 = 3:1$ [Externally]
$(x, y) = ?$

Now, By using formula

$$(x, y) = \left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right) = \left(\frac{3 \times 9 - 1 \times 5}{3 - 1}, \frac{3 \times 6 - 1 \times (-2)}{3 - 1}\right)$$

$$= \left(\frac{27 - 5}{2}, \frac{18 + 2}{2}\right) = \left(\frac{22}{2}, \frac{20}{2}\right) = (11, 10)$$

Thus, the coordinates of the point that divides the line segment externally in the ratio 3:1 is (11, 10).

Example 3

In what ratio does the point (3, –2) divide the line segment joining the points (1, 4) and (–3, 16)?

Solution: Here,

$(x_1, y_1) = (1, 4)$
$(x_2, y_2) = (-3, 16)$
$(x, y) = (3, -2)$
$m_1 : m_2 = ?$

Now, $x = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}$

or, $3 = \dfrac{m_1(-3) + m_2(1)}{m_1 + m_2}$

or, $3m_1 + 3m_2 = -3m_1 + m_2$

or, $6m_1 = -2m_2$

or, $\dfrac{m_1}{m_2} = \dfrac{-2}{6} = \dfrac{-1}{3}$

$\therefore m_1 : m_2 = 1 : -3$

Point (3, –2) divides the line segment joining the points (1, 4) and (–3, 16) in the ratio of 1:3 externally.

Thought Provoking Question: How do you know that the line divides the segment externally?

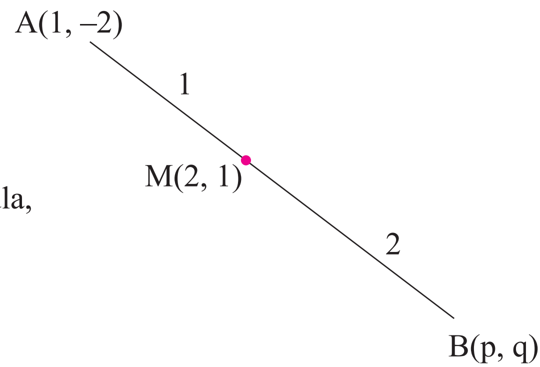
Is the question solved by taking $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$? Discuss with friends and solve it. Is the answer same or different? Check it.

Example 4

If the point (2, 1) divides the line segment joining the points (1, –2) and (p, q) in the ratio of 1:2, find the value of (p, q).

Solution: Here,

Here, point M(2, 1) divides the line segment joining the points A(1, –2) and B (p, q) in the ratio of 1:2.



$(x, y) = (2, 1)$

$(x_1, y_1) = (1, -2)$

$(x_2, y_2) = (p, q)$

$m_1 : m_2 = 1:2$

Now, according to section formula,

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

or, $2 = \frac{1 \times p + 2 \times 1}{1 + 2}$

or, $2 = \frac{p + 2}{3}$

or, $p + 2 = 6$

$\therefore p = 4$

or, $y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$

or, $1 = \frac{1 \times q + 2 \times (-2)}{1 + 2}$

or, $1 = \frac{q - 4}{3}$

or, $q - 4 = 3$

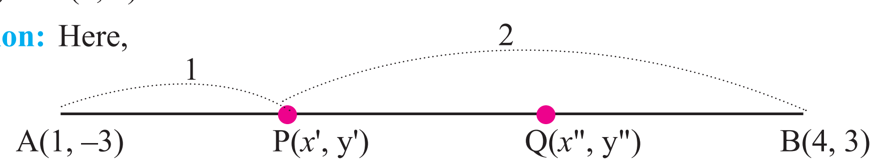
$\therefore q = 7$

Thus, $(p, q) = (4, 7)$

Example 5

Find the coordinates of the points which trisect the line segment joining the points (1, –3) and (4, 3).

Solution: Here,



Let P (x', y') and Q (x'', y'') divide the line segment AB joining the points A (1, –3) and B (4, 3) in three equal parts.

For point P, $(x_1, y_1) = (1, -3)$

$(x_2, y_2) = (4, 3)$

$m_1 : m_2 = AP : BP = AP : (PQ + BQ) = AP : (AP + AP) = AP : 2AP = 1 : 2$

$(x, y) = (x', y') = ?$

Now, $(x', y') = \left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right) = \left(\frac{1\times4+2\times1}{1+2}, \frac{1\times3+2\times(-3)}{1+2}\right)$

$= \left(\frac{4+2}{3}, \frac{3-6}{3}\right) = \left(\frac{6}{3}, \frac{-3}{3}\right) = (2, -1)$

$\therefore (x', y') = (2, -1)$

Then, Q be the midpoint of PB

$(x_1, y_1) = (2, -1)$

$(x_2, y_2) = (4, 3)$

Midpoint $(x'', y'') = ?$

Now, $(x'', y'') = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{2+4}{2}, \frac{-1+3}{2}\right) = \left(\frac{6}{2}, \frac{2}{2}\right) = (3,1)$

Thus, (2, –1) and (3, 1) are the points that divide the line segment AB joining the points A (1, –3) and B (4, 3) in three equal parts.

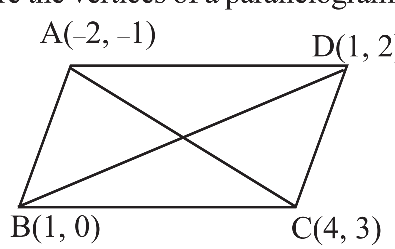
Just as P divides AB in the ratio 1:2, does Q divide AB in the ratio 2:1? If it does, find the coordinates of Q using the Section Formula, similar to P. Discuss the solution among friends, solve it, and check whether the answer is the same or different.

Example 6

Prove that the points (–2, –1), (1, 0), (4, 3) and (1,2) are the vertices of a parallelogram.

Solution: Here,

A(–2, –1), B(1, 0), C(4, 3) and D(1, 2) are the vertices of quadrilateral ABCD.



Now, the coordinates of the midpoint of diagonal AC

$(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-2+4}{2}, \frac{-1+3}{2}\right) = \left(\frac{2}{2}, \frac{2}{2}\right) = (1,1)$

Similarly, the coordinates of the midpoint of diagonal BD

$(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{1+1}{2}, \frac{0+2}{2}\right) = \left(\frac{2}{2}, \frac{2}{2}\right) = (1,1)$

Since the coordinates of the midpoints of both diagonals are the same, so quadrilateral ABCD is a parallelogram. Thus, the given points are the vertices of a parallelogram.

Exercise 3.1 (B)

1. Write the section formula of the straight line.
2. State the special cases of the section formula with suitable examples.
3. Find the coordinates of the point dividing the line joining the given points internally in the given ratio:
   a. (6, –10) and (–4, 14) in the ratio 3:4
   b. (3, 5) and (–2, –7) in the ratio 3:2
   c. (4, 3) and (6, 3) in the ratio 2:5
4. Find the coordinates of the point dividing the line joining the given points externally in the given ratio:
   a. (–3, 2) and (6, 5) in the ratio 2:1
   b. (–3, 2) and (4, –4) in the ratio 4:3
   c. (3, –2) and (–3, –4) in the ratio 1:2
5. In what ratio does the point (–2, 2) divide the line joining (4, 6) and $\left(\frac{1}{2}, -3\right)$? Find it.
6. In what ratio does the point (1, 3) divide the line joining (4, 6) and (3, 5)? Find it.
7. In what ratio does the point $\left(\frac{1}{2}, \frac{-3}{2}\right)$ divide the line joining (3, –5) and (–7, 9)? Find it.
8. Find the midpoint of the line joining (–3, –6) and (1, –2).
9. The midpoint of the line joining M(1, 4) and N(x', y') is (–2, 2). Find the value of (x', y').
10. If the midpoint of the line segment is (4, 3) and the point of one end is (0, 2), find the point of other end.
11. If A(2, –1), B(–1, 4), and C(–2, 2) are the vertices of ΔABC, find the mid points of each side.
12. Find the coordinates of the points that divide the line joining the given points into three equal parts:
    a. A(1, –3) and B(4, 3)
    b. P(1, –2) and Q(–3, 4)
    c. M(–5, –5) and N(25, 10)
13. In what radio does X- axis divides the line joining (2, 3) and (5, 6)? Find it.
14. In what radio does Y- axis divides the line joining (–4, 5) and (3, –7)? Find it.

15. If the point (3, y) lies on the line segment joining the points (7, –3) and (–2, –5), find the value of y.
16. If the point (x, –5) lies on the line segment joining the points (2, 3) and (–6, 5), find the value of x.
17. Prove that the given points are the vertices of a parallelogram.

    a. (1, 2), (3, 0), (7, 4) and (5, 6) b. (–1, 0) (3, 1) (2, 2) and (–2, 1)

    c. (3, –2) (4, 0) (6, –3) and (5, –5)
18. If the three given points are the three consecutive vertices of a parallelogram, find the coordinates of the fourth vertex.

    a. A(2, 3), B(4, –1) and C(0,5) b. A(2, 6), B(6, 2) and C(12, 4)

    c. (1, 2), (3, 1) and (5, 3)
19. If the points (1, 2), (3, 0), (x, 4) and (5, y) are the vertices of a parallelogram, find the values of x and y.
20. The diagonals of a parallelogram with vertices (3, 2) and (5, 10) intersect at the point (3, 4). Find the remaining two vertices.
21. If A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D (x_4, y_4) are the coordinates of the vertices of a parallelogram ABCD, prove that: $(x_1 + x_3) = (x_2 + x_4)$ and $(y_1 + y_3) = (y_2 + y_4)$.

Answer

1 - 2. Show to the teacher. 3.a. $\left(\frac{12}{7}, \frac{2}{7}\right)$ b. $\left(0, \frac{-11}{5}\right)$ c. $\left(\frac{32}{7}, 3\right)$

4. a. (15, 8) b. (25, –22) c. (9, 0) 5. 4:5 6. –3:2 7. 1:3

8. (–1, –4) 9. (–5, 0) 10. (8, 4)

11. Midpoint of AB = $\left(\frac{1}{2}, \frac{3}{2}\right)$, Midpoint of BC = $\left(\frac{-3}{2}, 3\right)$ and Midpoint of CA = $\left(0, \frac{1}{2}\right)$

12. a. (2, –1) and (3, 1) b. $\left(\frac{-1}{3}, 0\right)$ and $\left(\frac{-5}{3}, 0\right)$ c. (5, 0) and (15, 5) 13. 1:–2

14. 4:3 15. $\frac{-35}{9}$ 16. 59 17. Show to the teacher.

18. a. (–2, 9) b. (8, 8) c. (3, 4) 19. $x = 7, y = 6$ 20. (3, 6) and (1, –2)

3.1.3 Slope of a Straight Line, Concept of x- intercept and y- intercept

Straight Line

Do the following activity to develop the concept of a straight line:

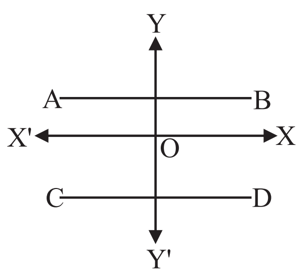
Activity 1

Represent the equation $3x - 2y - 4 = 0$ on a graph. For this, take suitable values of x and y by yourself. What type of figure is formed? Write your conclusion.

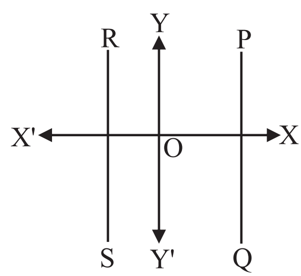
When the equation $3x - 2y - 4 = 0$ is represented on a graph, it is observed that an equation of degree 1 represents a straight line.
The equation of a straight line is also called a linear equation.

In general, straight lines are of three types: Lines parallel to the X- axis, lines parallel to the Y- axis, and lines making an angle with the X- axis.

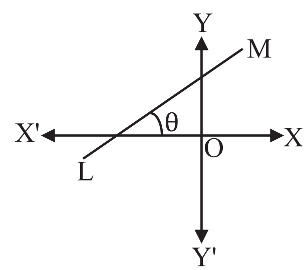
These can be seen in the figures given below.



Lines parallel to the X- axis, Horizontal straight line

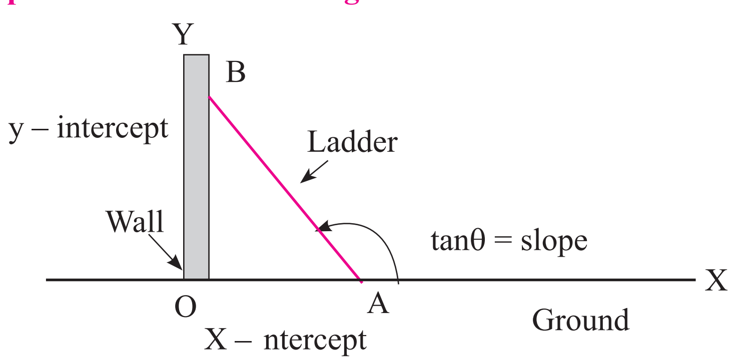


Lines parallel to the Y- axis Vertical straight line



Making the angle to the X- axis

A. Slope or Gradient of a Straight Line



Have you ever seen a ladder placed at your home? How is the ladder positioned? Observe and analyze.

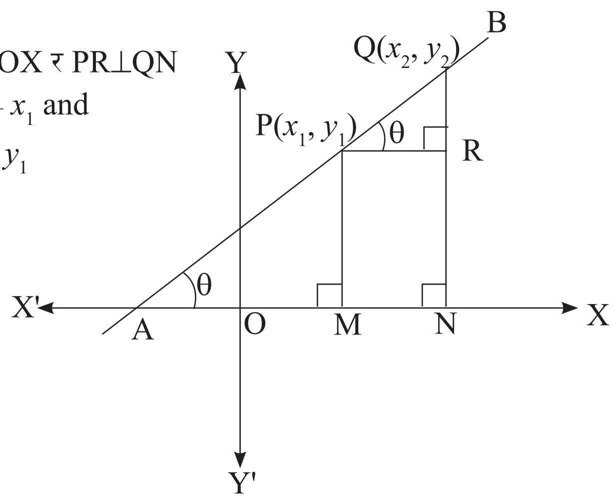
The ladder must have been placed leaning against the wall, tilted or inclined. The extent to which the ladder is tilted or inclined while placing it is called the inclination of the ladder with the ground.

OA = the part made by line AB with the X-axis, called the x- intercept, and

OB = the part made by line AB with the Y-axis, called the y- intercept.

B. Slope of the Line Joining the Points (x_1, y_1) and (x_2, y_2)

Let the line PQ, which passes through the points P (x_1, y_1) and Q (x_2, y_2) intersect the X- axis at point A and form an angle of $\angle PAX = \theta$ in the positive direction with the X- axis.



In the figure, draw PM⊥OX, QN⊥OX र PR⊥QN

Then, $PR = MN = ON - OM = x_2 - x_1$ and

$QR = QN - RN = QN - PM = y_2 - y_1$

with $\angle QPR = \angle PAX = \theta$

In the right-angled triangle ΔPRQ,

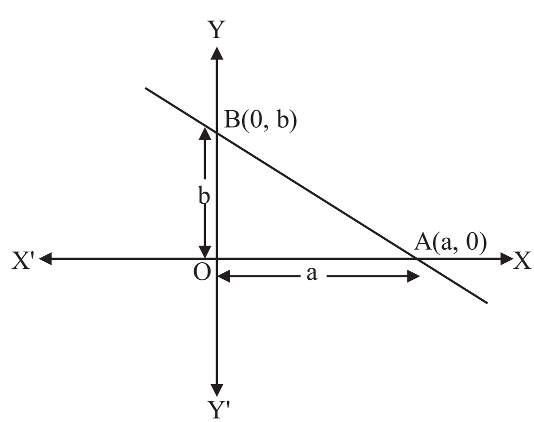
∴ The inclination (or slope) of line PQ(m) = tanθ

$$= \frac{QR}{PR}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

C. Calculation of x- intercept and y- intercept

How do you find the x- intercept and y- intercept of a straight line? Let's discuss.



In the adjacent figure, the straight line AB intersects the X- axis at point A(a, 0) and the Y- axis at point B(0, b). This means that OA = a and OB = b. These segments are respectively called the x- intercept and the y- intercept.

∴ x- intercept = a, y- intercept = b.

Example 1

Find the slope of straight lines which makes an angle of 45° with the X- axis.

Solution: Here,

$\theta = 45°$

Slope of straight line (m) = $\tan\theta = \tan 45° = 1$

Example 2

Find the slope of line joining the points (4, 5) and (6, 7).

Solution: Here,

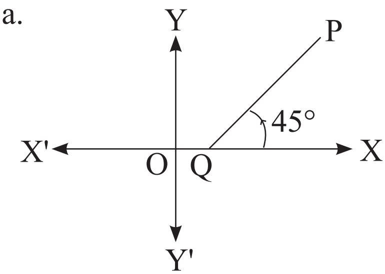
$(x_1, y_1) = (4, 5)$ and $(x_2, y_2) = (6, 7)$

Thus, slope (m) = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{7-5}{6-4} = \frac{2}{2} = 1$

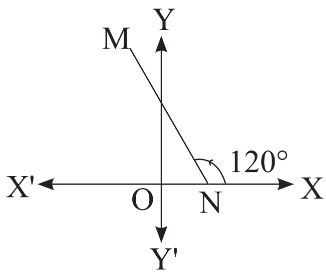
Exercise 3.1 (C)

1. Write the definition of the slope of straight line.
2. If the angle made by the straight line with the positive direction of the X - axis is θ, write the formula to find the slope.
3. What is the slope of the line joining the points (x_1, y_1) and (x_2, y_2)?
4. Define the x- intercept and the y- intercept of a straight line.
5. Find the slope of straight line which makes an angle of 60° with the X- axis.
6. Find the slope of straight line which makes an angle of 135° with the X- axis.
7. Find the slopes of the given line segments.

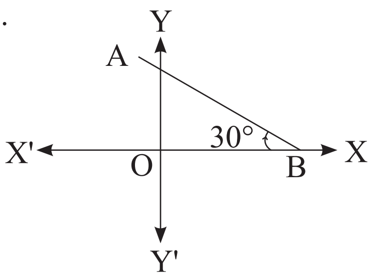
a.



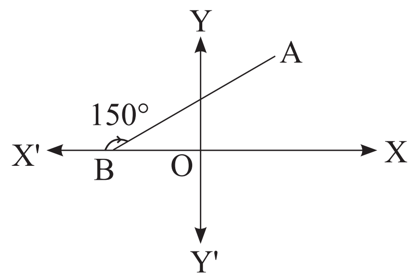
b.



c.



d.



8. Find the slope of line joining the points (–4, 3) and (2, –5).
9. Find the slope of line joining the points (0, –5) and (3, 0).
10. a. If the slope of the line joining the points (2, 5) and (6, *p*) is $\frac{-1}{2}$, find the value of *p*.
    b. If the slope of the line joining the points (*p*, 8) and (3, 4) is 2, find the value of *p*.
11. If the slope of the line joining the points P(8, 6) and Q(4, 2) and slope of the line joining the points A(7, 9) and B(*p*, 3) are equal, find the value of *p*.
12. If the points P(–2, –2), (1, 1) and (m, 2) are collinear, find the value of m.

Answer

1 - 4. Show to the teacher. 5. $\sqrt{3}$ 6. –1 7. a. 1 b. $-\sqrt{3}$

c. $\frac{-1}{\sqrt{3}}$ d. $\frac{1}{\sqrt{3}}$ 8. $-\frac{4}{3}$ 9. $\frac{5}{3}$ 10. a. 5 b. 5 11. 1 12. 2

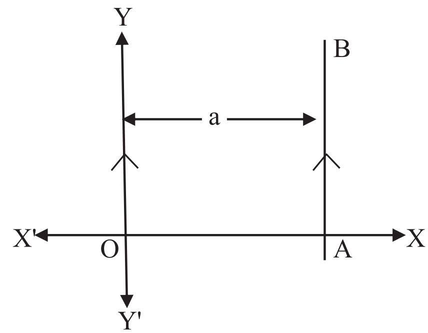
3.1.4 Equations of Straight Line

1. Equation of line parallel to Y- axis

In the adjoining figure, the line AB is parallel to the Y- axis, and the distance from the Y- axis to AB is a units.

Thus, the x- coordinate of every point lying on AB is a.

Therefore, $x = a$ is the equation of the line AB.



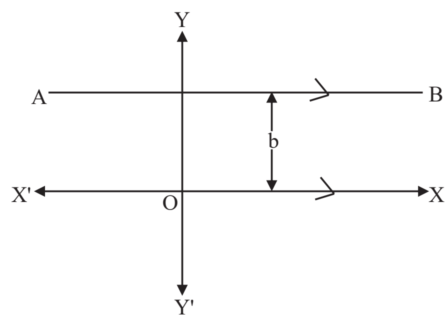
If the line AB is to the left side of the Y- axis and the distance from the Y- axis to the line is a units, what would its equation be? Discuss this among your friends and present it in the classroom.

2. Equation of line parallel to X – axis

In the adjoining figure, the line AB is parallel to the X- axis, and the distance from the X- axis to AB is b units.

Thus, the y- coordinate of every point lying on AB is b.

Therefore, $y = b$ is the equation of the line AB.

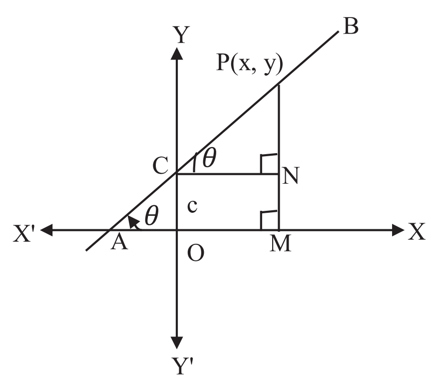


If the line AB is below the X-axis and the distance from the X-axis to the line is b units, what would its equation be? Discuss this among your friends and present it in the classroom.

3. Equation of straight line in slope intercept form

In the adjoining figure, the straight line AB makes an angle $\angle BAX = \theta$ with the positive direction of the X-axis.

Its y- intercept is OC = (c) From any point P(x, y) lying on this line, we draw PM $\perp$ OX. We also draw CN $\perp$ PM. We have $\angle PCN = \angle BAX = \theta$, OM = x and PM = y.



In right angled triangle PNC,

$\tan\theta = \frac{PN}{CN}$

or, $m = \frac{PM - NM}{CN}$

or, $m = \frac{PM - OC}{OM}$

or, $m = \frac{y - c}{x}$

or, $y - c = mx$

$\therefore y = mx + c$

Alternative method

The coordinate of point C = (0, *c*)

Slope of AB = Slope of CP (why? Discuss.)

or, $m = \frac{y - c}{x - 0}$

or, $mx = y - c$

$\therefore y = mx + c$ is the required equation.

a. If the line AB passes through origin, $c = 0$ and its equation be $y = mx + 0$

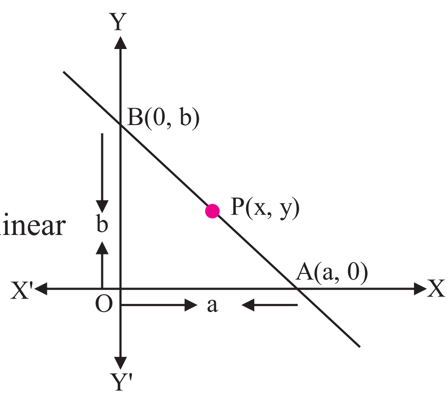
$\therefore y = mx$

b. If the line AB is parallel to X- axis, $m = \tan 0° = 0$, and the equation of line is $y = mx + c$

Or, $y = 0.x + c$ $\therefore y = c$

4. Equation of straight line in double intercepts form

In the adjoining figure, the straight line AB intersects the X- axis at the point A(*a*, 0) and the Y- axis at the point B(0, *b*). Therefore, OA = a and OB = *b*.



Let P(*x*, y) be any point lying on AB.

Slope of AP = $\frac{y - 0}{x - a} = \frac{y}{x - a}$

Slope of PB = $\frac{b - y}{0 - x} = \frac{b - y}{0 - x}$

The points A(*a*, 0), P(*x*, y), and B(0, *b*) are collinear (lie on the same line).

Therefore, Slope of AP = Slope of PB

Or, $\frac{y}{x - a} = \frac{b - y}{0 - x}$

Or, $-xy = bx - ab - xy + ay$

Or, $bx + ay = ab$

Dividing both sides by *ab*:

$$\frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1$$

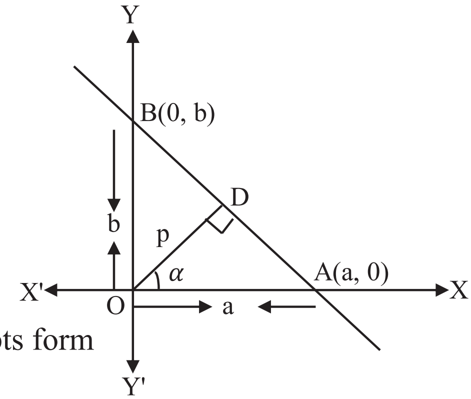
This is the equation of the line AB in the double intercepts form.

5. Equation of straight line in perpendicular form

In the given figure, the straight line AB intersects the X- axis at the point A(a, 0) and the Y- axis at the point B(0, b). Therefore, OA = a and OB = b.

A perpendicular line OD ⊥ AB is drawn, where OD = p and ∠DOA = a.

In the right-angled triangle Δ ODB, ∠BOD = 90° – α and ∠OBD = 180° – (90° – α) – 90° = α.



$$\sin\alpha = \frac{OD}{OB} = \frac{p}{b}$$

$$\text{or, } b = \frac{p}{\sin\alpha}$$

In the right-angled triangle ΔODA

$$\cos\alpha = \frac{OD}{OA} = \frac{p}{a}$$

$$\text{or, } a = \frac{p}{\cos\alpha}$$

Now, the equation of line AB from the intercepts form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\text{or, } \frac{x}{\frac{p}{\cos\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

$$\text{or, } \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

Thus, $x\cos\alpha + y\sin\alpha = p$ is the required equation of line AB.

Example 1

Find the equation of the line that is at a distance of 6 units to the right of the Y- axis, and is parallel to the Y- axis. Also, show it on a graph.

Solution

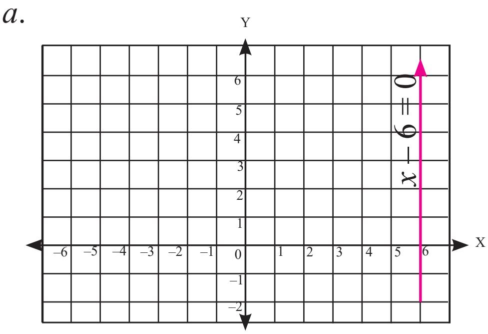
Here, the equation of a line parallel to the Y- axis is $x = a$.

For a line that lies 6 units to the right of the Y- axis is $a = 6$.

Thus, the required equation of the line is $x = a$.

Or, $x = 6$

$\therefore\ x - 6 = 0$



Example 2

Find the equation of the straight line that is parallel to the X- axis and passes through the point (–2, 5).

Solution

Here, the equation of a line parallel to the X- axis is $y = b$.

Since, this line passes through the point (–2, 5), we have $b = 5$.

Therefore, the required equation is $y = 5$.

Or, $y – 5 = 0$ is the required equation.

Example 3

Find the slope and y- intercept of the straight line $2x – 10y = 8$.

Solution

Here, $2x – 10y = 8$

Or, $2x – 8 = 10y$

Or, $y = \frac{2x}{10} - \frac{8}{10}$

Comparing this equation with the slope-intercept form, $y = mx + c$,

Slope $(m) = \frac{1}{5}$

y- intercept $(c) = \frac{-4}{5}$

Example 4

Find the equation of the straight line that intersects the Y- axis with y- intercept 3, and makes an angle of 60° with the X- axis.

Solution

Here, the y- intercept is $(c) = 3$.

The slope is $(m) = \tan\theta = \tan 60° = \sqrt{3}$

Now, using the equation of a straight line in the slope-intercept form, $y = mx + c$

$\therefore \sqrt{3}\,x – y + 3 = 0$ is the required equation of the line.

Example 5

Find the equation of the straight line having x- intercept = 3 and y- intercept = –4.

Solution

Here, x- intercept (a) = 3

y- intercept (b) = –4

The equation of a line in the intercept form is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Or, $\frac{x}{3} + \frac{y}{-4} = 1$

Or, $\frac{4x - 3y}{12} = 1$

$\therefore$ $4x - 3y = 12$ is the required equation of the line.

Example 6

Find the x- intercept and y- intercept of the straight line $3x + 4y = 24$.

Solution

Here, $3x + 4y = 24$

Dividing both sides by 24,

Or, $\frac{3x}{24} + \frac{4y}{24} = \frac{24}{24}$

Or, $\frac{x}{8} + \frac{y}{6} = 1$

Comparing it with $\frac{x}{a} + \frac{y}{b} = 1$

Thus, x- intercept (a) = 8 and y- intercept (b) = 6

Example 7

Find the equation of the straight line that makes intercepts of equal magnitude but opposite signs on the axes, and passes through the point (3, –4).

Solution

Here, if the x- intercept is (a) = k, then the y- intercept must be (b) = $-k$

Now, using the equation of a straight line in the intercept form: $\frac{x}{a} + \frac{y}{b} = 1$

Or, $\frac{x}{k} + \frac{y}{-k} = 1$

Or, $\frac{x - y}{k} = 1$

Or, $x - y = k$ ……..(i)

Since the line passes through the point (3, –4), we substitute $x = 3$ and $y = –4$ into equation i. $3 – (–4) = k$.

Or, $3 + 4 = k$

$\therefore k = 7$

Substituting the value of k in equation (i)

$x – y = 7$

$\therefore x – y – 7 = 0$ is the required equation of the line.

Example 8

If the perpendicular distance from the origin to a certain line is 4 units and this perpendicular makes an angle of 60° with the X- axis, find the equation of the line.

Solution: Here,

Perpendicular distance $(p) = 4$

Angle $(\alpha) = 60°$

Now, using the Normal form (Perpendicular form): $x\cos\alpha + y\sin\alpha = p$

Or, $x\cos 60° + y\sin 60° = 4$

or, $x.\frac{1}{2} + y.\frac{\sqrt{3}}{2} = 4$

or, $\frac{x+\sqrt{3}y}{2} = 4$

or, $x + \sqrt{3}\,y = 8$

$\therefore x + \sqrt{3}\,y - 8 = 0$

This is the required equation of the line.

Exercise 3.1 (D)

1. Write the equation of the straight line that is parallel to the X - axis.
2. Write the equation of the straight line that is parallel to the Y- axis.
3. Write the equation of the straight line in slope-intercept form.
4. Write the equation of the straight line in double -intercept form.
5. Write the equation of the straight line in normal form (perpendicular form).
6. Define p and α in the equation of the straight line in normal form, $x\cos\alpha + y\sin\alpha = p$.
7. What are the three standard forms of the equation of straight line? Write the equations for those forms.
8. Find the equation of the line that is at a distance of 3 units to the left of the Y- axis and is parallel to the Y- axis. Also, present it graphically.
9. Find the equation of the line that is at a distance of –3 units from the X - axis and is parallel to the X- axis. Also, present it graphically.

10. Find the equation of the straight line that is parallel to the X- axis and passes through the point (–3, 2)
11. Find the equation of the straight line that is parallel to the Y- axis and passes through the point (5, –2).
12. Find the slope and y- intercept of the straight lines given by the following equations:

    a. $y = 5x - 2$ b. $y = -2x + 4$ c. $y = 12x$ d. $y = 6$

    e. $y = \frac{1}{2}x - \frac{2}{3}$ f. $x + y + 1 = 0$ g. $2y - 10x = 8$ h. $y = \sqrt{3}x$
13. Find the equation of the straight line in the following conditions:

    a. Slope = 5 and y- intercept = 3 b. Slope = –2 and y- intercept = –1

    c. Slope = tan 120° and y- intercept = –5

    d. Slope = tan (–60°) and y- intercept = 3

    e. Slope = 3 and passes through the origin

    f. Slope = $\frac{1}{3}$ and passes through the point (0, 1)

    g. Passes through the origin and makes an angle of 45° with the X- axis.

    h. Passes through the origin and makes an angle of 150° with the X- axis.
14. Find the equation of the straight line in the following conditions:

    a. x- intercept = 3 and y- intercept = – 4

    b. x- intercept = –2 and y- intercept = 3

    c. x- intercept = $\frac{3}{5}$ and y- intercept = $\frac{9}{2}$
15. Find the x- intercept and y- intercept of the lines given by the following equations:

    a. $4x - 3y - 12 = 0$ b. $5x + 3y + 15 = 0$
16. Find the equation of the straight line in the following conditions:

    a. Passes through the point (2, –1) and makes equal intercepts on the axes.

    b. Passes through the point (3, 4) and makes equal intercepts on the axes.

    c. Makes intercepts of equal magnitude but opposite in signs on the axes and passes through the point (2, 3).

    d. Makes intercepts of equal magnitude but opposite in signs on the axes and passes through the point (6, –5).

    e. Cuts the axes such that the x- intercept is double the y- intercept, and passes through the point (3, 2).

    f. Passes through the point (3, 4) and the sum of its x- intercept and y- intercept is 15.

17. If a straight line passes through the point (2, 3) and this point bisects the segment of the line intercepted between the axes, find the equation of that line.

18. Line AB intersects the X- axis at A(6, 0) and the Y- axis at B(0, 8).

    a. Find the x- intercept and y- intercept of the line AB.

    b. Find the equation of the line AB.

    c. Find the length of AB.

19. Find the equation of the line that passes through the point (–5, 8) and whose y- intercept is double the x- intercept. Also, prove that the same line passes through (–1, 0).

20. If a line passes through the point (2, 3) and this point divides the segment of the line intercepted between the axes in the ratio 3:4, find the equation of that line.

21. Find the equation of the straight line in the following conditions:

    a. $p = 4$ units and $\alpha = 30°$ b. $p = 2$ units and $\alpha = 90°$

    c. $p = 1$ units and $\alpha = -60°$ d. $p = \frac{5}{7}$ units and $\alpha = 135°$

    e. $p = 13$ units and $\alpha = 45°$ f. $p = \sqrt{8}$ units and $\alpha = 150°$

Here, p is the perpendicular distance from the origin to the line, α is the angle made by the perpendicular with the positive X- axis.

Answer

1-7. Show to the teacher. 8. $x - 3 = 0$ 9. $y + 3 = 0$

10. $y - 2 = 0$ 11. $x - 5 = 0$ 12. a. 5, –2 b. –2, 4 c. 12, 0

d. 0, 6 e. $\frac{1}{2}, \frac{-2}{3}$ f. –1, –1 g. 5, 4 h. $\sqrt{3}$, 0

13. a. $5x - y + 3 = 0$ b. $2x + y + 1 = 0$ c. $\sqrt{3}x + y + 5 = 0$

d. $\sqrt{3}x + y - 3 = 0$ e. $3x - y = 0$ f. $x - 3y + 3 = 0$

g $x - y = 0$ h. $x + \sqrt{3}y = 0$

14. a. $4x - 3y - 12 = 0$ b. $3x - 2y + 6 = 0$ c. $15x + 2y - 9 = 0$

15. a. 3, –4 b. –3, –5 16. a. $x + y - 1 = 0$

b. $x + y - 7 = 0$ c. $x - y + 1 = 0$ d. $x - y - 11 = 0$

e. $x + 2y - 7 = 0$ f. $2x + y - 10 = 0$, $2x + 3y - 18 = 0$

17. $3x + 2y - 12 = 0$ 18. a. 6, 8 b. $4x + 3y - 24 = 0$

c. 10 19. $2x + y + 2 = 0$ 20. $7x + 10y - 70 = 0$

21. a. $\sqrt{3}x + y - 8 = 0$ b. $y - 2 = 0$ c. $x - \sqrt{3}y - 2 = 0$

d. $7x - 7y + 5\sqrt{2} = 0$ e. $x + y - 13\sqrt{2} = 0$ f. $\sqrt{3}x - y + 4\sqrt{2} = 0$

3.1.5 Reduction of Ax + By + C = 0 in the Standard Forms

The general equation of a line expressed in first degree is $Ax + By + C = 0$, where x, y are variables and A, B, C are constants. This is the general form of the equation of a straight line. How can this equation be converted into slope-intercept form, intercept form, and perpendicular (normal) form? Discuss among friends and present in the classroom.

A. Reduction in slope intercept form

Here, $Ax + By + C = 0$

Or, $By = -Ax - C$ Or, $y = \frac{-A}{B}x - \frac{C}{B}$

Comparing it with $y = mx + c$,

$$\text{Slope } (m) = \frac{-A}{B} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y}$$

$$y - \text{intercept } (c) = \frac{-C}{B} = \frac{-\text{ Constant}}{\text{Coefficient of } y}$$

B. Reduction in intercepts form

Here, $Ax + By + C = 0$

Or, $Ax + By = -C$

Dividing both sides by $-C$

$$\frac{Ax}{-C} + \frac{By}{-C} = \frac{-C}{-C} \qquad \text{Or, } \frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1$$

Comparing it with $\frac{x}{a} + \frac{y}{b} = 1$

$$x\text{- intercept (a)} = \frac{-C}{A} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y}$$

$$y\text{- intercept (b)} = \frac{-C}{B} = \frac{-\text{ Constant}}{\text{Coefficient of } y}$$

C. Reduction in perpendicular form

Here, multiplying the equation $Ax + By + C = 0$ by k gives $k(Ax + By + C) = 0$. Assuming that this equation is identically equal to $x\cos\alpha + y\sin\alpha - p = 0$, when k has some fixed value.

$x\cos\alpha + y\sin\alpha - p = k(Ax + By + C)$ ….. (i)

Or, $x\cos\alpha + y\sin\alpha - p = xkA + ykB + kC$

Assuming $k(Ax + By + C) = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ are identical

$$\cos\alpha = kA \;.....(ii) \qquad \text{and} \quad \sin\alpha = kB \;.....(iii)$$

Now, adding equation (ii) and (iii) in square form,

$$\cos^2\alpha + \sin^2\alpha = k^2(A^2 + B^2)$$

Or, $1 = k^2(A^2 + B^2)$

Or, $k^2 = \dfrac{1}{A^2 + B^2}$

$$\therefore\ k = \pm\frac{1}{\sqrt{A^2 + B^2}}$$

Substitute value of k in equation (i),

$$x\cos\alpha + y\sin\alpha - p = \pm\frac{1}{\sqrt{A^2+B^2}}(Ax + By + C)$$

$$\text{or, } x\cos\alpha + y\sin\alpha - p = \frac{Ax}{\pm\sqrt{A^2+B^2}} + \frac{By}{\pm\sqrt{A^2+B^2}} + \frac{C}{\pm\sqrt{A^2+B^2}}$$

Comparing,

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2+B^2}}, \qquad \sin\alpha = \frac{B}{\pm\sqrt{A^2+B^2}} \text{ र } \qquad p = \frac{C}{\pm\sqrt{A^2+B^2}}$$

Here, since p represents the perpendicular distance, the sign (+) or (–) should be taken so that its value becomes positive.

Note: The equation of a straight line of the first degree is often written in the form $ax + by + c = 0$ instead of $Ax + By + C = 0$ as well.

Example

Convert the equation $\sqrt{3}x - y + 2 = 0$ into slope-intercept form, intercept form, and normal form.

Solution

To convert into slope-intercept form,

Here, $\sqrt{3}x - y + 2 = 0$

Or, $\sqrt{3}x + 2 = y$

Compare $y = \sqrt{3}x + 2$ with $y = mx + c$

Slope (m) = $\sqrt{3}$, and y- intercept $(c) = 2$

Alternative method

Compare $\sqrt{3}x - y + 2 = 0$ with $Ax + By + C = 0$

$$A = \sqrt{3},\ B = -1 \text{ र } C = 2$$

$$\text{Slope (m)} = \frac{-A}{B} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\text{y - intercept (c)} = \frac{-C}{B} = \frac{-\text{Constant}}{\text{Coefficient of } y} = \frac{-2}{-1} = 2$$

To convert into intercept form,

Here, $\sqrt{3}x - y + 2 = 0$ or, $\sqrt{3}x - y = -2$

Dividing both side by – 2,

$$\frac{\sqrt{3}x}{-2} - \frac{y}{-2} = \frac{-2}{-2} \quad \text{or, } \frac{x}{\frac{-2}{\sqrt{3}}} +$$

Comparing it with $\frac{x}{a} + \frac{y}{b} = 1$

x – intercept $(a) = \frac{-2}{\sqrt{3}}$

y – intercept $(b) = 2$

Alternative method

Here, comparing $\sqrt{3}x - y + 2 = 0$ with $Ax + By + C = 0$

$A = \sqrt{3}$, $B = -1$ and $C = 2$

$$X\text{– intercept } (a) = \frac{-C}{A} = \frac{-\text{ constant}}{\text{coefficient of } x} = \frac{-2}{\sqrt{3}}$$

$$Y\text{– intercept } (b) = \frac{-C}{B} = \frac{-\text{ constant}}{\text{coefficient of } y} = \frac{-2}{-1} = 2$$

To convert into perpendicular form

Comparing, $\sqrt{3}x - y + 2 = 0$ with $Ax + By + C = 0$

$A = \sqrt{3}, B = -1$ र $C = 2$

$p = \frac{C}{-\sqrt{A^2+B^2}}$ [Here, p is perpendicular distance which is always positive, we take -ve sign.)

$$= \frac{2}{-\sqrt{(\sqrt{3})^2+(-1)^2}} = \frac{2}{-\sqrt{3+1}} = \frac{2}{-\sqrt{4}} = \frac{2}{-2} = -1$$

$$\cos\alpha = \frac{A}{-\sqrt{A^2+B^2}} = \frac{\sqrt{3}}{-\sqrt{(\sqrt{3})^2+(-1)^2}} = \frac{\sqrt{3}}{-\sqrt{3+1}} = \frac{\sqrt{3}}{-\sqrt{4}} = -\frac{\sqrt{3}}{2}$$

$$\sin\alpha = \frac{B}{-\sqrt{A^2+B^2}} = \frac{-1}{-\sqrt{(\sqrt{3})^2+(-1)^2}} = \frac{-1}{-\sqrt{3+1}} = \frac{-1}{-\sqrt{4}} = \frac{1}{2}$$

Now, $x\cos\alpha + y\sin\alpha + p = 0$ substituion values of $\cos\alpha$ and $\sin\alpha$

$x\left(-\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) + (-1) = 0$ [Here, $\cos\alpha$ is negative and $\sin\alpha$ is positive, so α lies in 2nd quadrant.]

So, $x\cos150^0 + y\sin150^0 - 1 = 0$

$\therefore x\cos150^0 + y\sin150^0 = 1$ is the required equation in perpendicular form.

Exercise 3.1 (E)

1. Convert the following equations into slope-intercept form.

   a. $\sqrt{3}x + 2y = 7$ b. $x - y = 6$ c. $3x - 2y + 8 = 0$

2. Convert the following equations into slope-intercept form.

   a. $3x + 4y = 12$ b. $2x - 3y - 6 = 0$ c. $\sqrt{3}x - 7y = 5$

3. Convert the following equations into perpendicular form.

   a. $x - y + 4 = 0$ b. $x = y - 2\sqrt{2}$ c. $3x + 4y = 15$

   d. $x + \sqrt{3y} = -4$ e. $5x = 12y + 13$ f. $\sqrt[x]{3} + y = 4$

4. The equation of the line is $x - y + 4 = 0$.

   a. Convert the given equation into the perpendicular (normal) form.

   b. What is the distance of that line from the origin? Find it.

5. The equation of the straight line is $4x + 3y - 24 = 0$.

   a. Present this equation of straight line in the intercept form.

   b. In which points the line intersects the X- axis and the Y- axis? Find it.

6. The equation of the line is $6x + 8y = 24$.

   a. Convert the given equation into the intercepts form.

   b. Find the values of x- intercept and the y- intercept.

   c. Find the ratio of x- intercept and the y- intercept.

7. The equation of the straight line is $3x + 4y = 12$.

   a. Convert the given equation into the slope-intercept form.

   b. Find the slope and the y- intercept of the line.

Answer

1. a. $y = \frac{-\sqrt{3}}{2x} + \frac{7}{2}$ b. $y = x - 6$ c. $y = \frac{3}{2}x + 4$

2. a. $\frac{x}{4} + \frac{y}{3} = 1$ b. $\frac{x}{3} + \frac{y}{-2} = 1$ c. $\frac{x}{\frac{5}{\sqrt{3}}} + \frac{y}{\frac{-5}{7}} = 1$

3. a. $x \cos 135° + y \sin 135° = 2\sqrt{2}$ b. $x \cos 135° + y \sin 135° = 2$

   c. $\frac{3}{5}x + \frac{4}{5}y = 3$ d. $x \cos 240° + y \sin 240° = 2$ e. $\frac{5x}{13} - \frac{12y}{13} = 1$

   f. $x \cos 60° + y \sin 60° = 2\sqrt{3}$

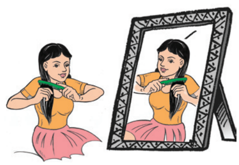
4. a. $x \cos 135° + y \sin 135° = 2\sqrt{2}$ b. $2\sqrt{2}$ 5.a. $\frac{x}{6} + \frac{y}{8} = 1$ b. (6, 0), (0, 8)

6. a. $\frac{x}{4} + \frac{y}{3} = 1$ b. 4, 3 c. 4:3

7. a. $y = \frac{-3}{4}x + 3$ b. $\frac{-3}{4}$, 3

3.2 Transformation

Observe the figure and discuss the relationship of transformation.



When observing the above pictures, the image of an object is seen in the mirror. In the clock, the minute hand moves from 0 minutes to 90° clockwise in 15 minutes. You place your book on a corner of a table and then move the book to another corner.

Transformation brings about a specific change in the position or size or both of a certain geometrical object on some definite rule.

Types of Transformation

When a closed door is opened, its position is changed. Is the size of the door also changed?

When a balloon is inflated, its size increases. Thus, after a transformation, the position or size or both of the image may change compared to the original figure.

Based on this, transformations are divided into two types: Isometric and non - isometric.

a. **Isometric Transformation:** A transformation in which only the position of the image changes compared to the original figure, but the shape and size remains the same, is called an isometric transformation. In this type, the object and its image are congruent. Reflection, rotation, and translation fall under this category.

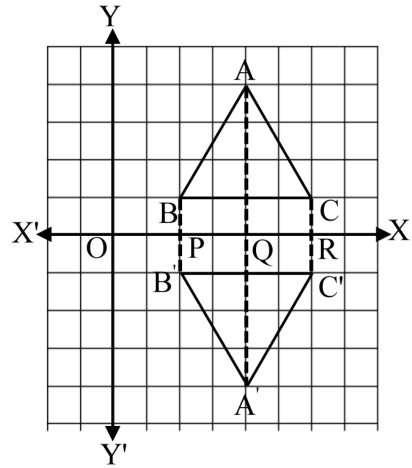
b. **Non-isometric Transformation:** A transformation in which the size of the image changes compared to the original figure is called non-isometric transformation. Enlargement falls under this category.

3.2.2 Reflection of Geometrical Shapes

Each student measures AQ, A'Q, BP, B'P and CR, C'R and ∠AQX, ∠BPX and ∠CRX from the given figure and discuss whether they obtained the same result.

AQ = A'Q = 4 units BP = B'P = 1 unit and CR, = C'R = 1 unit. That is, the object and its image are at equal distances from the axis of reflection. Also, ∠AQX = 90°, ∠BPX = 90° and ∠CRX = 90°. Also, AA'⊥OX, BB'⊥ OX and CC'⊥OX. That is the line joining the object and its image is perpendicular to the axis of reflection.

The corresponding sides and corresponding angles of the object and its reflected image are equal to each other. Therefore, ΔABC ≅ ΔA'B'C'. That is, the object and the images formed after reflecting that object are congruent.



Observing the graph given alongside,

$A(4, 4) \xrightarrow{\text{Re:X- axis}} A'(4, -4)$

$B(2, 1) \xrightarrow{\text{Re:X- axis}} B'(2, -1)$

$C(6, 1) \xrightarrow{\text{Re:X- axis}} C'(6, -1)$

Thus, $P(x, y) \xrightarrow{\text{Re:X- axis}} P'(x, -y)$

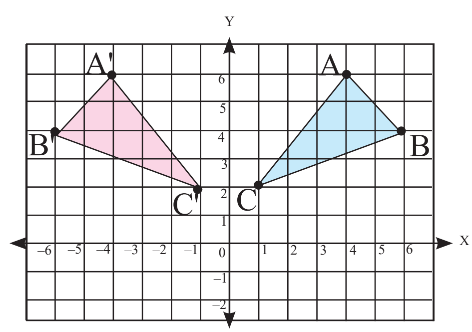
> When triangle ΔABC is reflected in the X- axis, after reflection the x- coordinates of vertices A, B, and C remain the same while the signs of their y- coordinates change. A point lying on the X- axis remains unchanged when reflected in the X- axis.

From the above discussion, we can write the following properties of reflection:

1. The object and its image are at equal distances from the axis of reflection.
2. The line joining the object and its image is perpendicular to the axis of reflection.
3. The object and its reflected image are congruent.
4. The object and its image are inverted.
5. Points lying on the given axis of reflection remain invariant (unchanged).

Activity 1

Reflect the triangle ΔABC on the graph across the Y- axis. Write down the coordinates of the image points A', B' and C'. Now, compare the coordinates of ΔABC and its image ΔA'B'C' and draw a conclusion.



Procedure: Take graph and draw the given triangle ΔABC. Reflect it across the Y- axis on the same drawing. Write the coordinates of the reflected triangle ΔA'B'C' obtained by this reflection and present the drawing. Then compare the coordinates of ΔABC and its image Δ A'B'C' and present your conclusion.

Conclusion: $P(x, y) \xrightarrow{\text{Re:Y- axis}} P'(-x, y)$

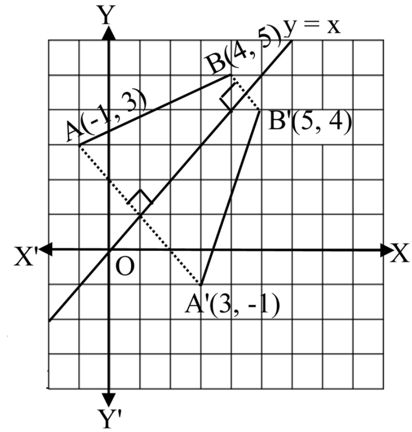
a. Reflection on the line $y = x$

What kind of line is $y = x$? Does this line pass through the origin? Also, through which quadrants does it pass? Discuss among your friends and present in the classroom.

$y = x$ is a line in which each point has x and y coordinates same, such as (0, 0), (1, 1), (–2, –2), etc. This line passes through the origin and goes through the first and third quadrants.

Activity 2

In the given graph, when the points of line AB with coordinates (–1, 3) and (4, 5) are reflected in the line $y = x$, the reflected points A' and B' of line A'B' are shown in the graph.



What are the coordinates of points A' and B'? Compare and draw a conclusion.

Procedure: Draw the given line AB on the graph and reflect it about the line $y = x$. After reflection, if the coordinates of A′ are (3, –1) in the image A'B', then what will be the coordinates of B'? Write it. Then, compare the coordinates of triangles ΔABC and ΔA'B'C' and present your conclusion.

From the figure, $A(-1, 3) \xrightarrow{\text{Re: } y = x} A'(3, -1)$

$$B(4, 5) \xrightarrow{\text{Re: } y = x} B'(5, 4)$$

Here, the positions of the coordinates of the image are interchanged. Therefore, when a point P(x, y) is reflected in the line y = x, its image is P′(y, x).

Conclusion: When a point P(x, y) is reflected in the line y = x, the image is P′(y, x)

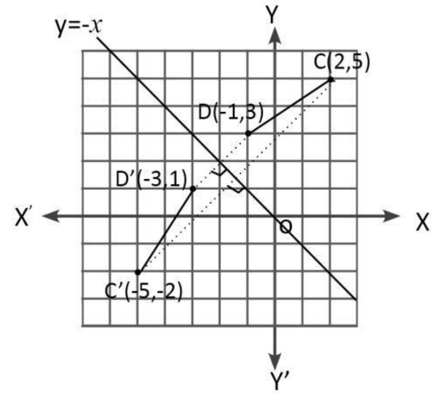
$$P(x, y) \xrightarrow{\text{Re: } y = x} P'(y, x)$$

b. Reflection on the line y = – x

What kind of line is $y = -x$? Does this line pass through the origin? Also, through which quadrants does it pass? Discuss among your friends and present in the classroom.

$y = -x$ is a line in which each point of x and y coordinates are equal in magnitude but opposite sign [such as (0, 0), (1, –1), (–2, 2)], etc. This line passes through the origin and goes through the second and fourth quadrants.

Activity 3



In the given graph, when the points of line CD with coordinates C(2, 5) and D(–1, 3) are reflected in the line $y = -x$, the reflected points C' and D' of line C'D' are shown in the graph.

What are the coordinates of points C' and D'? Compare and draw a conclusion.

Procedure: Reflect the points C(2, 5) and D(–1, 3) of line CD on the line $y = -x$, write the coordinates of the reflected points C' and D', and discuss whether everyone obtained the same result to confirm consistency.

Here, not only are the positions of the coordinates of the image interchanged, but their signs are also changed. Therefore, when a point P(x, y) is reflected in the line y = $-x$, its image is P′($-y$, $-x$).

$$C(2, 5) \xrightarrow{\text{Re: } y = -x} C'(-5, -2)$$

$$D(-1, 3) \xrightarrow{\text{Re: } y = -x} D'(-3, 1)$$

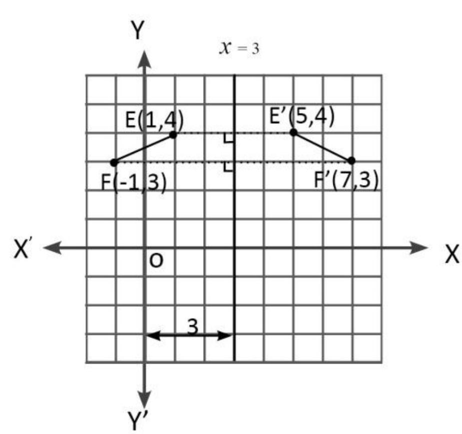
Conclusion: $P(x, y) \xrightarrow{\text{Re: } y = -x} P'(-y, -x)$

c. Reflection in the line $x = a$

What does $x = a$ represent? Which axis is this line parallel to? Discuss among your friends and present in the classroom.

The line $x = a$ is a line that is parallel to the Y- axis and at a distance of a units from it. That is, for every point on the line $x = a$, the y- coordinate is the same a. while the y- coordinate varies.

Activity 4



Problem: In the given graph, when the points, E(1, 4) and F(–1 ,3) of line EF are reflected across the line $x = 3$, the reflected points E' and F' of line E'F' are shown in the graph. What are the coordinates of E′ and F'? Compare and draw a conclusion.

Procedure: Reflect the points E(1, 4) and F(−1, 3) across the line $x = 3$, write the coordinates of the reflected points E′ and F′ of line EF and discuss on your conclusion.

$$E(1, 4) \xrightarrow{x = 3} E'(5, 4) = E'(2 \times 3 - 1, 4)$$

$$F(-1, 3) \xrightarrow{x = 3} F'(7, 3) = F'(2 \times 3 - (-1), 4)$$

Thus, $P(x, y) \xrightarrow{\text{Re: } x = a} P'(2a - x, y)$

d. Reflection in the line $y = b$

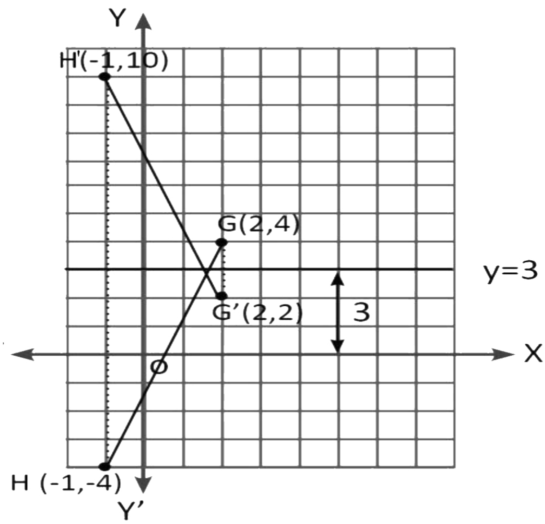
What does $y = b$ represent? Which axis is this line parallel to? Discuss among your friends and present in the classroom.

Solution

The line $y = b$ is a line that is parallel to the X- axis and at a distance of b units from it. That is, for every point on the line $y = b$, the x- coordinate is different and the y- coordinate is same b.

Activity 5

In the given graph, when the points, G(2, 4) and H(–1, –4) of line GH are reflected across the line $y = 3$, the reflected points G' and H' of line G'H' are shown in the graph. What are the coordinates of G' and H'? Compare and draw a conclusion.



Procedure: Reflect the points G(2, 4) and H(–1, –4) across the line y = 3, write the coordinates of the reflected points G' and H' of line G'H' and discuss whether everyone obtained the same result to confirm consistency.

$$G(2, 4) \xrightarrow{\text{Re: } y = 3} G'(2, 2) = G'(2, 2 \times 3 - 4)$$

$$H(-1, -4) \xrightarrow{\text{Re: } y = 3} H'(-1, 10) = H'(-1, 2 \times 3 - (-4))$$

Thus, $P(x, y) \xrightarrow{\text{Re: } y = b} P'(x, 2b - y)$

Example 1

The vertices of ΔABC are A(2, –1), B(–3, 0) and C(–4,–2). Reflect ΔABC across the line $y = x$ and write the coordinates of the vertices of the image ΔA'B'C'. Present both triangles on the graph.

Solution: Here,

The vertices of ΔABC are A(2, –1), B(−3, 0) and C(−4,−2).

To reflect ΔABC on the line $y = x$

We know that,

$$P(x, y) \xrightarrow{\text{Re: } y = x} P'(y, x)$$

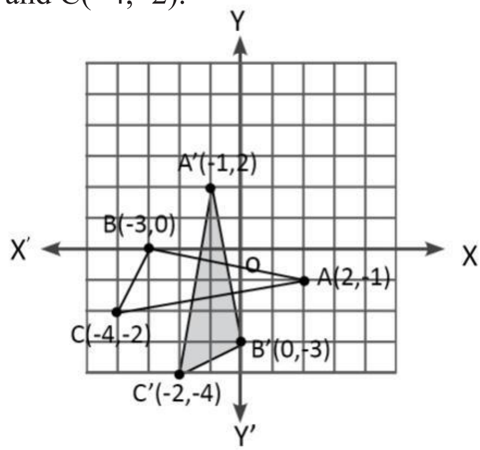
Thus, A(2, –1) → A'(–1, 2)

B(–3, 0) → B'(0, –3)

C(–4, –2) → C'(–2, –4)

Thus, the coordinates of image ΔA'B'C' has A'(–1, 2), B'(0, –3) and C'(–2, –4).

Thus, Δ ABC and image Δ A'B'C' are presented in the adjoining graph.



Example 2

The vertices of ΔABC are A(5, 3), B(−1, –2) and C(−3, 2).

a. Reflect ΔABC across the line y = $-x$ and find the coordinates of the vertices of the image ΔA′B′C′.

b. Again reflect Δ A'B'C' across the line $x = -3$ and find the coordinates of the vertices of the image ΔA"B"C". Present ΔABC, ΔA'B'C' and ΔA"B"C" on the same graph.

Solution: Here,

The vertices of ΔABC are A(5, 3), B(−1, –2) and C(−3, 2).

a. To reflect ΔABC on the line y = $-x$

We know that,

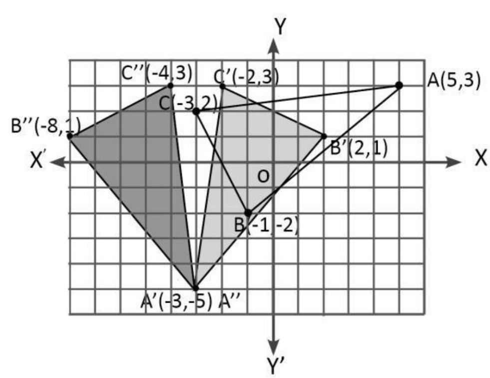
$$P(x, y) \xrightarrow{\text{Re: } y = -x} P'(-y, -x)$$

Thus,

$$A\,(5, 3) \xrightarrow{\text{Re: } y = -x} A'(-3, -5)$$

$$B\,(-1, -2) \xrightarrow{\text{Re: } y = -x} B'(2, 1)$$

$$C\,(-3, 2) \xrightarrow{\text{Re: } y = -x} C'(-2, 3)$$



Thus, the coordinates of image ΔA'B'C' has A'(–3, –5), B'(2, 1) and C'(–2, 3).

b. Again, to reflect ΔA'B'C' in the line $x = -3$

We know that,

$$P(x, y) \xrightarrow{\text{Re: } x = a} P'(2a - x, y)$$

Thus $A'(-3, -5) \xrightarrow{\text{Re: } x = -3} A''[2 \times (-3) - (-3), -5] = A''(-3, -5)$

$B'(2, 1) \xrightarrow{\text{Re: } x = -3} B''[2 \times (-3) - 2, 1] = B''(-8, 1)$

$C'(-2, 3) \xrightarrow{\text{Re: } x = -3} C''[2 \times (-3) - (-2), 3] = C''(-4, 3)$

Thus, the final coordinates of image ΔA"B"C" has A'(–3, –5), B'(–8, 1) and C'(–4, 3).

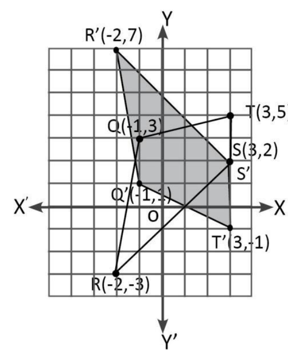
Again, ΔABC, ΔA'B'C' and ΔA″B″C″ are presented in the adjoining graph.

Example 3

If Q(–1, 3), R(–2, –3), S(3, 2) and T(3, 5) are the vertices of the quadrilateral QRST. Write the coordinates of the vertices of the image quadrilateral formed by reflecting quadrilateral QRST in the line y = 2, and present both quadrilaterals on a graph.

Solution: Here,

Q(–1, 3), R(–2, –3), S(3, 2) and T(3, 5) are the vertices of the quadrilateral QRST.



To reflect quadrilateral QRST on the line $y = 2$

We know that,

$$P(x, y) \xrightarrow{\text{Re: } y = b} P'(x, 2b - y)$$

Thus, $Q(-1, 3) \xrightarrow{\text{Re: } y = 2} Q'(-1, 2 \times 2 - 3) = Q'(-1, 1)$

$R(-2,-3) \xrightarrow{\text{Re: } y = 2} R'[-2, 2 \times 2 - (-3)] = R'(-2, 7)$

$S(3, 2) \xrightarrow{\text{Re: } y = 2} S'(3, 2 \times 2 - 2) = S'(3, 2)$

$T(3, 5) \xrightarrow{\text{Re: } y = 2} T'(3, 2 \times 2 - 5) = T'(3, -1)$

Hence vertices of the image quadrilateral Q'R'S'T' are Q'(–1, 1), R'(–2, 7), S'(3, 2) and T'(3, –1).

Again, quadrilateral QRST, and image quadrilateral Q'R'S'T' are presented on the graph.

Example 4

a. If reflection R_1 transforms A(3, 5) to A′(−3, 5), find the axis of reflection.

b. If reflection R_2 transforms B(5, –2) to B′(−2, 5), find the axis of reflection.

Solution: Here

a. To reflect R_1 transforms A(3, 5) to A′(−3, 5),

$$A(3, 5) \xrightarrow{R_1} A'(-3, 5)$$

So, $P(x, y) \xrightarrow{R_1} P'(-x, y)$

Since the coordinates of the given object and its image correspond to a | reflection on the Y- axis,

So, R_1 represents a reflection on the Y- axis.

b. To reflect R_2 transforms B(5, –2) to B′(−2, 5),

$$B(5, -2) \xrightarrow{R_2} B'(-2, 5)$$

So, $P(x, y) \xrightarrow{R_2} P'(y, x)$

Since the coordinates of the given object and its image correspond to a reflection on the $y = x$

So, R_2 represents a reflection on the line $y = x$.

Example 5

The vertices of the Δ ABC are A(2, 3), B(3, 2) and C(1, 1) respectively.

a. If the image of A(2, 3) is A'(2, –3), find the axis of reflection.

b. On the basis of the axis of reflection, find the coordinates of the remaining vertices.

c. Present Δ ABC and Δ A'B'C' on the same graph.

Solution: Here,

The vertices of the Δ ABC are A(2, 3), B(3, 2) and C(1, 1).

a. The image of A(2, 3) is A'(2, –3)

$$A(2, 3) \rightarrow A'(2, -3)$$

Thus, $P(x, y) \rightarrow P'(x, -y)$

Since the coordinates of the given object and its image correspond to a reflection on the X- axis

So, it represents a reflection on the X- axis.

b. To find the coordinates of the image of the remaining vertices B(3, 2) and C(1, 1)

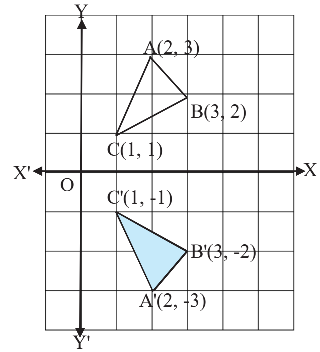
B (3, 2) → B'(3, –2)

C (1, 1) → C'(1, –1)

Thus, the coordinates of the image of the remaining vertices are B'(3, –2) and C'(1, –1) respectively.

c. To present the Δ ABC and Δ A'B'C',

Thus, Δ ABC and Δ A'B'C' are presented on the same graph.



Exercise 3.2 (A)

1. What is transformation? Write with examples.
2. Write the definition of reflection and mention its three properties.
3. Find the coordinates of the image formed by reflecting the given points A(1, 1), B(–3, 0), C(4, 2), D(–5, –3), and E(2, –3) on the following axes/lines:

   a. X- axis b. Y- axis c. Line $y = x$ d. Line $y = -x$

   e. Line $x = -3$ f. Line $y = 5$
4. If A(–2, 0), B(6, 2), and C(5, 3) are the vertices of ΔABC, then
   a. Find the coordinates of the vertices of the image ΔA'B'C' formed by reflecting ΔABC on the line $y = x$.
   b. Find the coordinates of the vertices of the image ΔA'B'C' formed by reflecting ΔABC on the line $x = 4$.
   c. Find the coordinates of the vertices of the image ΔA'B'C' formed by reflecting ΔABC on the line $y = -2$. Present the object (pre-image) and the image on a graph.
5. P(–1, 3), Q(–3, –1), R(3, –4), and S(2, 1) are the vertices of a quadrilateral PQRS:
   a. Find the coordinates of the vertices of the image quadrilateral formed by reflecting quadrilateral PQRS on the line $y = -x$, and present both the object and the image on a single graph.
   b. Find the coordinates of the vertices of the image quadrilateral formed by reflecting quadrilateral PQRS on the line $x = 1$, and present both the object and the image on a single graph.

6. Find the axis of reflection on the following conditions:

a. A(3, 5) → A'(–5, –3) b. B(2, –1) → B'(4, –1)

c. C(5, 7) → C'(–5, 7) d. D(–3, 4) →D'(–3, –8)

7. The vertices of ΔABC are A(2, 3), B(3, 2), and C(1, 1). Find the coordinates of the vertices of the image ΔA"B"C" formed by first reflecting ΔABC on the line $y = x$, and then reflecting the image ΔA'B'C' again on the line $x = 3$. Present all three triangles on a single graph.

8. ΔXYZ is reflected to form ΔX'Y'Z', where the vertices of ΔX'Y'Z' are X'(4, –2), Y'(8, –2), and Z'(8, 4). If one vertex of ΔXYZ is X(2, –4), find the remaining vertices. Also, find the axis of reflection.

9. A(2, 5), B(–3, 3), and C(1, –4) are the vertices of ΔABC. If the image of A(2, 5) is A'(2, –5) then:

a. Find the axis of reflection.

b. Find the coordinates of the images of the remaining vertices using the axis of reflection found above.

c. Present ΔABC and ΔA'B'C' on a single graph.

Answer

1-2. Show to the teacher. 3. a. A' (1,–1), B' (–3, 0), C' (4, –2) D' (–5, 3), E(2, 3)

b. A' (–1, 1), B' (3, 0), C'(–4, 2), D' (5, –3), E'(–2, –3)

c. A' (1, 1), B'(0, –3), C' (2, 4), D' (–3, –5), E' (–3, 2)

d. A' (–1, –1), B'(0, 3), C'(–2, –4), D'(3, 5), E'(3, –2)

e. A' (–7, 1), B' (–3, 0), C' (–10, 2), D'(–1, –3), E'(–8, –3)

f. A'(1, 9), B'(–3, 10), C' (4, 8), D'(–5, 13), E'(2, 13)

4. a. A'(0, –2), B'(2, 6), C'(3, 5) b. A'(10, 0), B' (2, 2), C'(3, 3)

c. A' (–2, –4), B' (6, –6), C'(5 – 7) Show the graph to the teacher.

5. a. P'(–3, 1), Q' (1, 3), R'(4, –3), S'(–1, –2)

b. P'(3, 3), Q' (5, –1), R'(–1, –4), S' (0, 1) Show the graph to the teacher.

6. a. $y = –x$ b. $x = 3$ c. y – axis d. $y = –2$

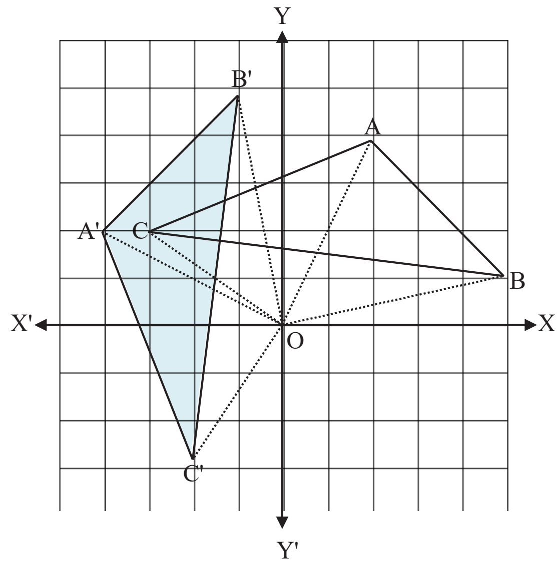
7. A' (–2, 3), B' (–3, 2), C' (–1, 1), A" (8, 3), B"(9, 2). C" (7, 1) Show the graph to the teacher.

8. Y(2, 8), Z(–4, –8), $y = –x$ 9. a. X– axis b. B'(–3, –3), C'(1, 4)

3.2.3 Rotation of Geometrical Shapes

a. Rotation through +90° about the origin (Positive quarter turn)

In the adjoining figure, ΔABC is rotated by +90° (positive direction, quarter turn) about the origin O(0, 0) to form the image ΔA'B'C'. Based on this figure, discuss with friends and present your conclusion for the following questions in the classroom:



a. Has ΔABC undergone an equal angular displacement in the same positive direction from every vertex?

b. Join AA', BB', and CC'. Draw the perpendicular bisectors of these three line segments. Do they all intersect at the same point?

c. Is ΔABC congruent or similar to ΔA'B'C'? Why?

d. Find the relationship between the coordinates of the vertices of ΔABC and its image ΔA'B'C'.

Solution: Here,

a. In the figure, ∠AOA' = 90°, ∠BOB' = 90° and ∠COC' = 90°. Therefore, ΔABC has undergone an equal angular displacement of 90° in the same positive direction from every vertex.

b. Join AA', BB', and CC' and draw the perpendicular bisectors of these three line segments. They all intersect at the origin O.

c. The corresponding sides and corresponding angles of ΔABC and its image ΔA'B'C' are equal to each other. Therefore, ΔABC ≅ ΔA'B'C'. That is, the object is congruent to its rotation image.

d. The coordinates of the vertices of ΔABC are A(2, 4), B(5, 1), and C(–3, 2). The coordinates of the vertices of the image ΔA'B'C' are A'(–4, 2), B'(–1, 5), and C'(–2, –3).

To the relationship of coordinates of the vertices,

$$A(2, 4) \xrightarrow{R_o [O, +90^\circ]} A'(-4, 2)$$

$$B(5, 1) \xrightarrow{R_o [O, +90^\circ]} B'(-1, 5)$$

$$C(-3, 2) \xrightarrow{(R_o [O, +90^\circ]} C'(-2, -3)$$

Here, the coordinates of X and Y are interchanged and the sign of X- coordinates is changed. Therefore, when P(x, y) is rotated by + 90° about O, the image is P'($-y$, x).

That is, $P(x, y) \xrightarrow{R_o: [O, +90^\circ]} P'(-y, x)$

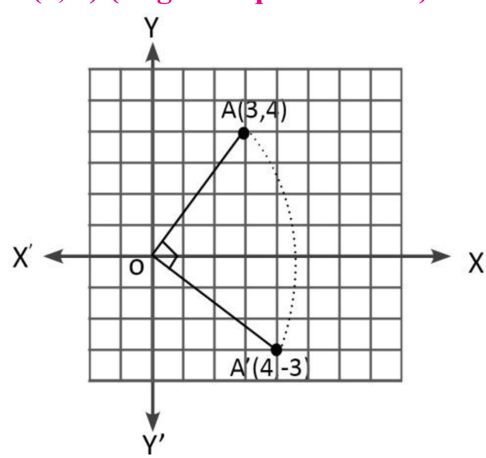
Thought Provoking Question: Why is the rotation of +90° same as the rotation of –270°? Discuss.

Based on the discussion above, the properties of rotation are as follows:

1. Every point of a geometric figure on a plane surface undergoes an equal angular displacement in the same direction during rotation.
2. The perpendicular bisector of the line segment joining a point on the figure and its corresponding image point passes through the center of rotation.
3. The object and its image after rotation are congruent.
4. Only the center of rotation remains an invariant point (unchanged point) during rotation.

b. Rotation through –90° about the origin (0, 0) (Negative quarter turn)

In the given figure, point A is rotated by –90° (a quarter turn in the negative direction) about the origin O(0, 0) to form the image A'. Write the coordinates of point A and its image A', and present the relationship between their coordinates in the box through discussion.



Here, the coordinates of point A are (3, 4) and the coordinates of its image A' are (4, –3).

To the relationship of coordinates of the vertices,

$$A(3, 4) \xrightarrow{R_o: [O, -90^\circ]} A'(4, -3)$$

Here, the coordinates of X and Y are interchanged and the sign of Y- coordinates is changed.

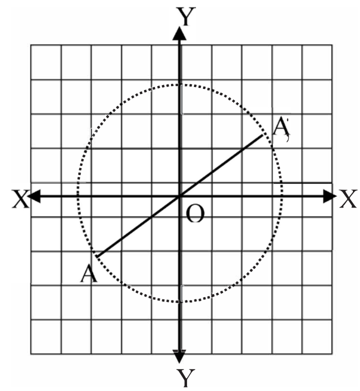
Therefore, when P(x, y) is rotated by –90° about O, the image is P'(y, –x).

That is, $P(x, y) \xrightarrow{R_o\,[O,\,-90°]} P'(y, -x)$

Why is the rotation of –90° same as the rotation of +270°? Discuss.

c. Rotation through 180° or Halfturn about the Origin

In the given figure, point A is rotated by 180° about the origin O(0, 0) to form the image A'. Write down the coordinates of point A and its image A', and discuss and present the relationship between the coordinates of A and A' in the classroom.



Here, the coordinates of point A are (3, 2) and the coordinates of its image A' are (–3, –2).

For the relationship between the coordinates of the vertices:

$A(3, 2) \xrightarrow{R_o:\,[O,\,180°]} A'(-3, -2)$

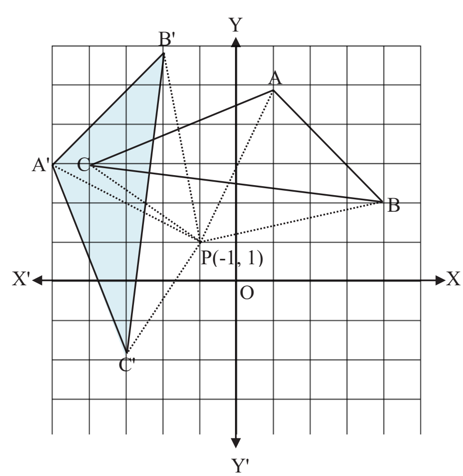
Here, the magnitude of the coordinates remains the same, but their signs are changed. Therefore, when P(x, y) is rotated by 180° about O, the image is P'(–x, –y).

That is, $P(x, y) \xrightarrow{R_o:\,[O,\,180°]} P'(-x, -y)$

Why is the rotation of +180° same as the rotation of –180°? Discuss.

d. Rotation through +90° about any point (a, b) (Positive quarter turn

In the figure on the side, ΔABC is rotated by + 90° (a positive quarter turn) about the point P(–1, 1) to form the image ΔA'B'C'. Discuss and present the relationship between the coordinates of the vertices of ΔABC and the vertices of its image ΔA'B'C' in the classroom.



Here, the coordinates of the vertices of ΔABC are A(1, 5), B(4, 2), and C(–4, 3). And, the coordinates of the vertices of ΔA'B'C' are A'(–5, 3), B'(–2, 6), and C(–3, –2).

For the relationship between the coordinates of the vertices:

$A(1, 5) \xrightarrow{Ro: [(-1, 1), +90^\circ]} A'(-5, 3) = A'[-5 + (-1 + 1), 1 - (-1 - 1)]$

$B(4, 2) \xrightarrow{Ro: [(-1, 1), +90^\circ]} B'(-2, 6) = B'[-2 + (-1 + 1), 4 - (-1 - 1)]$

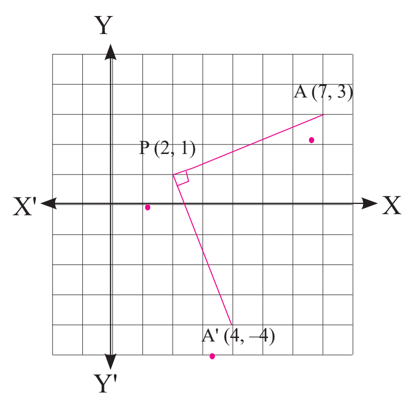
$C(-4, 3) \xrightarrow{Ro: [(-1, 1), +90^\circ]} C'(-3, -2) = C'[-3 + (-1 + 1), -4 - (-1, -1)]$

Here, the coordinates of x and y are interchanged and the sign of the x coordinate is changed. However, the x and y coordinates of the center of rotation are added to the x- coordinate of the image, while the x and y coordinates of the center of rotation are subtracted from the y- coordinate of the image. Therefore, when P(x, y) is rotated by a positive quarter turn about (a, b), the image is $P'[-y + (a + b), x - (a - b)]$.

That is, $P(x, y) \xrightarrow{[(a, b), +90^0]} P'[-y + (a + b), x - (a - b)]$

e. Rotation through –90° about any Point (a, b) (Negative quarter turn)

In the given figure, point A is rotated by –90° (a quarter turn in the negative direction) about the point P(2, 1) to form the image A'. Write the coordinates of A and its image A', and discuss and present the relationship between their coordinates in the classroom.



Here, the coordinates of point A are (7, 3), and the coordinates of its image A' are (4, –4).

For the relationship between the coordinates of the vertices:

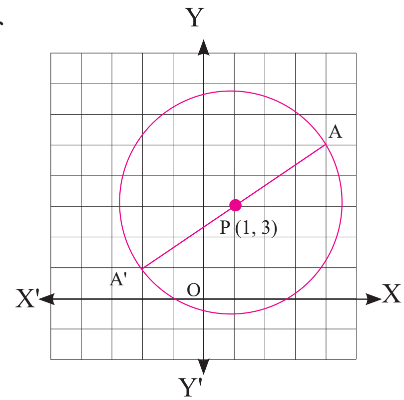
$A\,(7, 3) \xrightarrow[\text{Rotation}]{[(2, 1), -90^\circ]} A'(4, -4) = A'[3 + (2 - 1), -7 + (2 + 1)]$

Here, the x and y coordinates are interchanged, and the sign of the y- coordinate is changed. However, the x and y coordinates of the center of rotation are added to the x- coordinate of the image, while the x and y coordinates of the center of rotation are added to the negative x-coordinate of the image. Therefore, when P(x, y) is rotated by a negative quarter turn about (a, b), the image is $P'[y + (a - b), -x + (a + b)]$.

That is,
$P(x, y) \xrightarrow[\text{Rotation}]{[(a, b), -90^\circ]} P'[y + (a - b), -x + (a + b)]$

f. Rotation through 180° or Halfturn about any point (*a*, *b*)

In the given figure, point A is rotated by 180° (a half turn) about any point P(1, 3) to form the image A'. Write down the coordinates of point A and its image A', and discuss and present the relationship between the coordinates of A and A' with your friends in the classroom.



Here, the coordinates of point A are (4, 5), and the coordinates of its image A' are (–2, 1).

For the relationship between the coordinates of the vertices:

$$A(4, 5) \xrightarrow[\text{Rotation}]{[(1,\ 3),\ 180°]} A'(-2, 1) = A'(-4 + 2 \times 1, -5 + 2 \times 3)$$

Here, sign of *x* coordinate is changed and twice the x- coordinate of centre point is added. Similarly sign of y- coordinate is also changed and twice the y- coordinate of centre point is added to get image point. Therefore, when P(*x*, *y*) is rotated by 180° about (a, b), the image is P'(–*x* + 2*a*, –*y* + 2*b*).

That is, $P(x, y) \xrightarrow[\text{Rotation}]{[(a,\ b),180°]} P'(-x + 2a, -y + 2b)$

Example 1

Write the coordinates of the vertices of ΔA'B'C', which is the image formed after rotating ΔABC with vertices A(2, 4), B(5, 1), and C(–3, 2) by a quarter turn in the positive direction about the origin. Also, present ΔABC and ΔA'B'C' on the single graph.

Solution: Here,

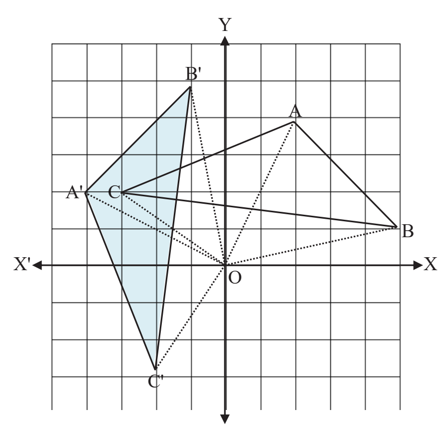
We know that

$$P(x, y) \xrightarrow{R_O: [O, +90°]} P'(-y, x)$$

Thus, $A(2, 4) \xrightarrow{R_O: [O, +90°]} A'(-4, 2)$

$$B(5, 1) \xrightarrow{R_O: [O, +90°]} B'(-1, 5)$$

$$C(-3, 2) \xrightarrow{R_O: [O, +90°]} C'(-2, -3)$$



Thus, A'(–4, 2), B'(–1, 5) and C'(–2, –3)
re the coordinates of the image Δ A'B'C'.

Thus, Δ ABC and Δ A'B'C' are presented on the same graph.

Example 2

The vertices of ΔPQR are P(–4, 6), Q(–1, –2), and R(3, –5). Find the coordinates of the vertices of the image of ΔP′Q′R′ formed when ΔPQR is rotated 90° clockwise about the origin, and represent both triangles in a graph.

Solution: Here,

The vertices of ΔPQR are P(–4, 6), Q(–1, –2), and R(3, –5).

To the ΔPQR is rotated 90° clockwise about the origin,

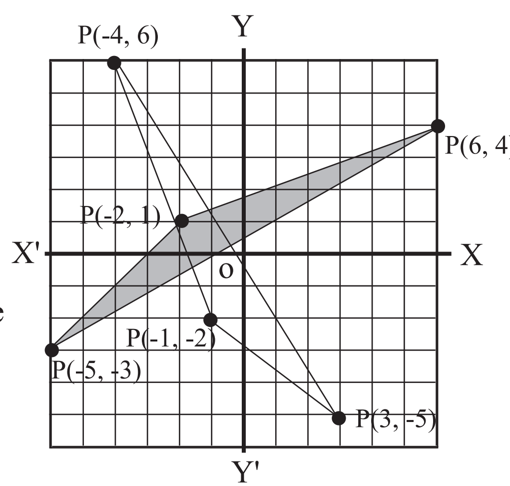
We know that

$$P(x, y) \xrightarrow{R_O: [O, -90°]} P'(y, -x)$$

Thus, $P(-4, 6) \xrightarrow{R_O: [O, -90°]} P'(6, 4)$

$Q(-1, -2) \xrightarrow{R_O: [O, -90°]} Q'(-2, 1)$

$R(3, -5) \xrightarrow{R_O: [O, -90°]} R'(-5, -3)$



So, P'(6, 4), Q'(–2, 1) and R'(–5, –3) are the coordinates of the image ΔP′Q′R′.

Thus, Δ PQR and Δ P'Q'R' are presented on the same graph.

Example 3

A(2, 1), B(1, –2), C(–3, –2) and D(–5, 1) are the vertices of trapezium ABCD. Find the coordinates of the vertices of the image of quadrilateral formed when ABCD is rotated 180° about the origin, and represent both quadrilaterals on a graph.

Solution: Here,

A(2, 1), B(1, –2), C(–3, –2) and D(–5, 1) are the vertices of trapezium ABCD.

ABCD is rotated through 180° about the origin, We know that

$$P(x, y) \xrightarrow{R_o: [O,\ 180°]} P'(-x, -y)$$

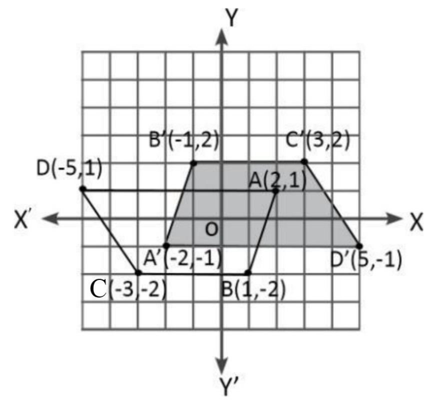
Thus,

$$A(2, 1) \xrightarrow{R_O: [O,\ 180°]} A'(-2, -1)$$

$$B(1, -2) \xrightarrow{R_O: [O,\ 180°]} B'(-1, 2)$$

$$C(-3, -2) \xrightarrow{R_O: [O,\ 180°]} C'(3, 2)$$

$$D(-5, 1) \xrightarrow{R_O: [O,\ 180°]} D'(5, -1)$$



So, A'(–2, –1), B'(–1, 2), C'(3, 2) and D'(5, –1) are the coordinates of the image trapezium A'B'C'D'.

Thus, trapeziums ABCD and A'B'C'D' are presented on the same graph.

Example 4

If the image of point A(–4, 3) rotated about the origin is A'(3, 4), find the angle and direction of the rotation.

Solution: Here,

The image A'(3, 4) is obtained by rotating point A(–4, 3) about the origin.

Therefore A(–4, 3) → A'(3, 4)

i.e., $P(x, y) \rightarrow P'(y, -x)$

This relationship indicates a negative quarter turn (–90°) rotation about the origin.

Example 5

ΔA'B'C' is obtained by reflecting ΔABC with vertices A(2, 3), B(1, 5), and C(–2, 4) on the line $y = x$. Furthermore, ΔA"B"C" is obtained by rotating ΔA'B'C' by a quarter turn in the positive direction about the origin. Find the coordinates of the vertices of ΔA'B'C' and ΔA"B"C", and present all three triangles on a graph.

Solution: Here,

A(2, 3), B(1, 5), and C(–2, 4) are the vertices of ΔABC. Now, ΔA'B'C' is formed by reflecting ΔABC in the line $y = x$.

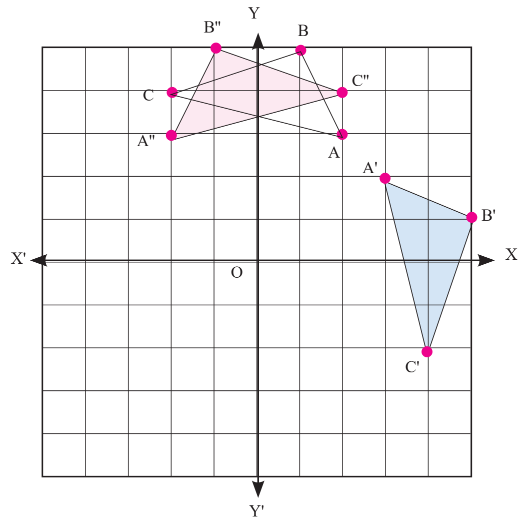
We know that, $P(x, y) \xrightarrow{Re: y = x} P'(y, x)$

So, $A(2, 3) \xrightarrow{Re: y = x} A'(3, 2)$

$B(1, 5) \xrightarrow{Re: y = x} B'(5, 1)$

$C(-2, 4) \xrightarrow{Re: y = x} C'(4, -2)$

Thus, when ΔABC is reflected in the line $y = x$, the coordinates of the image ΔA′B′C′ are A′(3, 2), B′(5, 1), and C′(4, –2). Again, when ΔA′B′C′ is rotated through quarter turn (90°) anticlockwise about the origin, the new image ΔA"B"C" is formed.



We know that,

$P(x, y) \xrightarrow{R_{O:} [O, +90°]} P'(-y, x)$

So, $A'(3, 2) \xrightarrow{R_{o:} [O, +90°]} A''(-2, 3)$

$B'(5, 1) \xrightarrow{R_{o:} [O, +90°]} B''(-1, 5)$

$C'(4, -2) \xrightarrow{R_{o:} [O, +90°]} C''(2, 4)$

Thus, the coordinates of the image ΔA"B"C", obtained by reflecting ΔA'B'C' on the line $y = x$, are A"(–2, 3), B"(–1, 5), and C"(2, 4).

Now, ΔABC, ΔA'B'C', and ΔA"B"C" are presented on the graph paper.

Exercise 3.2 (B)

1. Define rotation and write down its properties.
2. Find the coordinates of the image obtained by rotating any point $P(x, y)$ by +90° about the origin and then again by 180° about the origin.
3. Find the coordinates of the image obtained by rotating any point $P(x, y)$ by 90° about the point (a, b) and then again by 180°.
4. Find the coordinates of the vertices of the image obtained by rotating the points P(7, 5), Q(–3, 4), R(–1, –3), S(6, –3), and T(–4, 7) about the origin in the following conditions:

   a. +90° b. –90° c. 180°

5. Find the coordinates of the vertices of the image obtained by rotating the points P(7, 5), Q(–3, 4), R(–1, –3), S(6, –3), and T(–4, 7) about the point (2, 1) in the following conditions:

   a. +90°   b. –90°   c. 180°

6. ΔABC has vertices A(1, 0), B(4, 5), and C(7, –2). Find the coordinates of the vertices of the image of ΔABC in the following conditions:

   a. If ΔABC is rotated by a quarter turn in the clockwise direction about the origin.

   b. If ΔABC is rotated by 180° about the origin.

   c. If ΔABC is rotated by +90° about the origin.

7. A(3, 7), B(1, –1), and C(6, 8) are the vertices of ΔABC. Find the coordinates of the vertices of the image triangle obtained by rotating ΔABC by a quarter turn in the positive direction about the origin, and present both triangles on a graph.

8. A(2, 1), B(5, 1), C(4, 4), and D(1, 4) are the vertices of a parallelogram ABCD. Find the coordinates of the vertices of the image obtained by rotating ABCD by –90° about the origin, and present both parallelograms on a graph.

9. A(3, 7), B(1, –1), and C(6, 8) are the vertices of ΔABC. Find the coordinates of the vertices of the image triangle obtained by rotating ΔABC by a quarter turn in the negative direction about the point (–1, 1), and present both triangles on a graph.

10. A(2, 1), B(5, 1), C(4, 4), and D(1, 4) are the vertices of a parallelogram ABCD. Find the coordinates of the vertices of the image obtained by rotating ABCD by –90° about the point (2, 3), and present both parallelograms on a graph.

11. If R_1, R_2, R_3, and R_4 denotes the rotation about the origin, then find the angle and direction of rotation in the following conditions:

    a. $A(-3, 4) \xrightarrow{R_1} A'(3, -4)$ b. $B(4, 5) \xrightarrow{R_2} B'(-5, 4)$

    c. $C(-1, -2) \xrightarrow{R_3} C'(-2, 1)$ c. $D(6, -7) \xrightarrow{R_4} D'(6, -7)$

12. A(5, 2), B(3, 1), and C(2, -4) are the vertices of ΔABC. ΔA'B'C' is formed by rotating ΔABC by + 90° about the origin, and ΔA"B"C" is formed by rotating ΔA'B'C' by 180° about the origin. Find the coordinates of the vertices of ΔA'B'C' and ΔA"B"C" and present on a graph.

13. ΔPQR has vertices P(3, 4), Q(–2, 6), and R(1, –5). ΔP'Q'R' is the image formed by reflecting ΔPQR in the line x = –2. ΔP"Q"R" is formed by rotating ΔP'Q'R' by 270° about the origin. Find the coordinates of the vertices of ΔP'Q'R' and ΔP"Q"R", and present all three triangles on a single graph.

Answer

1 - 3. Show to the teacher. 4. a. P'(–5, 7), Q'(–4, –3), R'(3, –1), S'(3, 6), T'(–7, –4)

b. P'(5,–7), Q'(4, 3), R'(–3, 1), S'(–3, –6), T'(7, 4)

c. P'(–7,–5), Q'(3, –4), R'(1, 3), S'(–6, 3), T'(4, –7)

5. a. P'(–2, 6), Q'(–1, –4), R'(6, –2), S'(6, 5), T'(–4, –5)

b. P'(6, –4), Q'(5, 6), R'(–2, 4), S'(–2, –3), T'(8, 7)

c. P'(–3, –3), Q'(7, –2), R'(5, 5), S'(–2, 5), T'(8, –5)

6. a. A'(0, –1), B'(5, –4), C'(–2, –7) b. A'(–1, 0), B'(–4, –5), C'(–7, 2)

c. A'(0, 1), B'(–5, 4), C'(2, 7) 7. A'(–3, –7), B' (–1, 1), C'(–6, –8)

8. A'(1, –2), B'(1, –5), C' (4, –4), D'(4, –1) 9. A'(–5, –5), B'(–3, 3), C'(–8, –6)

10. A'(0, 3), B'(0, 0), C'(3, 1), D'(3, 4) 11. a. 180° b. +90° c. –90° d. 360°

12. A' (–2, 5), B'(–1, 3), C'(4, 2), A"(2, –5), B"(1, –3), C"(–4, –2)

13. P'(–7, 4), Q'(–2, 6), R'(–5, –5), P"(4, 7), Q"(6, 2), R"(–5, 5)

3.2.4 Enlargement and Reduction of Geometrical Shapes

In the graph ΔABC with vertices A(1, 1), B(5, 2), C(2, 4) is enlarged by a scale factor of 2 with the center of enlargement at the origin O. Discuss the following questions with your classmates and present them in the classroom:

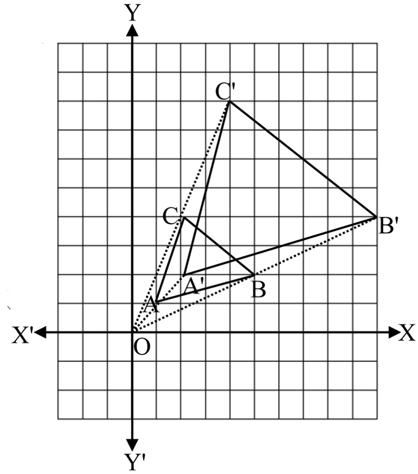
a. Measure OA, OA', OB , OB', OC and OC'. Find the values of $\frac{OA}{OA'}$, $\frac{OB}{OB'}$ and $\frac{OC}{OC'}$ and determine their relationships.

b. Are A and A', B and B', C and C' in the same direction from the center of enlargement O?

c. Are ΔABC and ΔA'B'C' similar or congruent? Write the reason.

d. Is ΔA'B'C' the image of ΔABC enlarged by twice its size?

a. In the given graph, $OA = \frac{1}{2}$ OA', $OB = \frac{1}{2}$ OB' and $OC = \frac{1}{2}$ OC'. Also $\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \frac{1}{2}$. Here, ΔA'B'C' is the image of ΔABC enlarged twice its size. The number 2 is called the scale factor (k).



Scale factor $(k) = \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = 2$

Or, scale factor $(k) = \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = 2$

Thus, ΔABC is enlarged with O as the center, and the point O is called the center of enlargement. The point A, its image A', and the center O lie on the same straight line.

The change in the size of a geometrical figure based on a fixed center of enlargement and a scale factor is called enlargement.

b. A and A', B and B', C and C' are in the same direction from the center of enlargement O. The figure and its image lie on the same direction from the center. The image and object are on the same direction and image is same orientation as the object. Therefore, the scale factor k. is positive.

c. ΔABC and ΔA'B'C' are similar because the corresponding sides are equal in the proportion. Therefore, the object and its image under enlargement are similar.

d. ΔA'B'C' is the image of ΔABC enlarged by twice its size.

In the graph below, ΔABC with vertices A(2, 2), B(10, 4), C(4, 8) is enlarged by a scale factor of $-\frac{1}{2}$ with the center of enlargement at the origin O. Discuss the answer of following questions among classmates and present in the classroom.

a. Measure OA, OA', OB, OB', OC and OC'. Find the values of $\frac{OA}{OA'}$, $\frac{OB}{OB'}$ and $\frac{OC}{OC'}$ and determine their relationships.

b. Are A and A', B and B', C and C' in the same direction from the center of enlargement O?

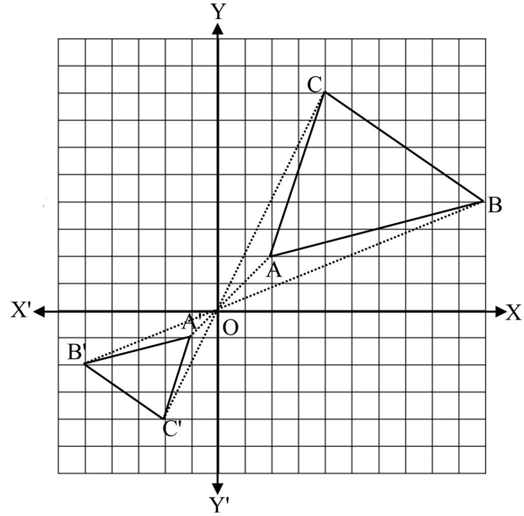
c. Are ΔABC and ΔA'B'C' similar or congruent? Write the reason.

d. Is ΔA'B'C' the image of ΔABC enlarged by twice its size?

a. In the graph alongside, If OA = 2OA', OB = 2OB', OC = 2OC', then $\frac{OA}{OA'}$, $\frac{OB}{OB'}$ and $\frac{OC}{OC'}$ = 2. Here, ΔA'B'C' is the image of ΔABC reduced by $\frac{1}{2}$ its size. The number $\frac{1}{2}$ is called the scale factor (k). "Scale factor" $k = \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{1}{2}$

Or, scale factor of enlargement

$(k) = \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{1}{2}$



Here, ΔABC is reduced with O as the center, and O is called the center of reduction (contraction). The point A, its image A', and the center O lie on the same straight line.

b. A and A', B and B', C and C' are in the opposite direction from the center of reduction O. The figure and its image lie in opposite directions from the center, and the image is inverted. Therefore, the scale factor k. is negative. Hence, $(k) = -\frac{1}{2}$

c. ΔABC and ΔA'B'C' are similar because the corresponding sides are equal in the proportion. Therefore, the figure and its image under enlargement or reduction are similar.

Based on the above discussion, the properties of enlargement/reduction are below:

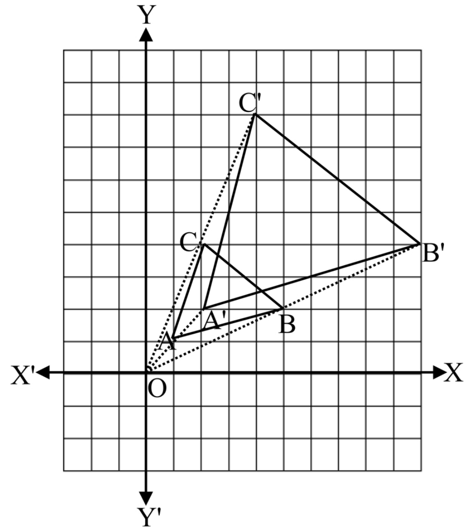
1. The figure and its image are always similar.
2. If the scale factor k is positive ($k > 0$), the figure and its image lie in the same direction from the center of enlargement.
   a. If $k > 1$, the image is enlarged.
   b. If $0 < k < 1$, the image is reduced.
3. If the scale factor k is negative ($k < 0$), the figure and its image lie in the opposite direction from the center, and the image is inverted.
   a. If $k < -1$, the image is enlarged.
   b. If $-1 < k < 0$, the image is reduced.
4. If $|k| = 1$, the object and its image are same.
5. The center of enlargement is an invariant point.

An enlargement with center O(0, 0) and scale factor k is often written as Enlargement E[O, k] or E[O(0, 0), k].

Use of Coordinates in Enlargement

a. When the centre of enlarement is at the origin O

From the graph alongside, write the coordinates of the vertices of ΔABC and its image ΔA'B'C'.



Discuss the relationships between the coordinates of the vertices of ΔABC and its image ΔA'B'C' with your classmates and present in the classroom.

The coordinates of vertices of ΔABC are A(1, 1), B(5, 2) and C(2, 4). Similarly, the coordinates of vertices of its image ΔA'B'C' are A'(2, 2), B'(10, 4) and C'(4, 8).

Again,

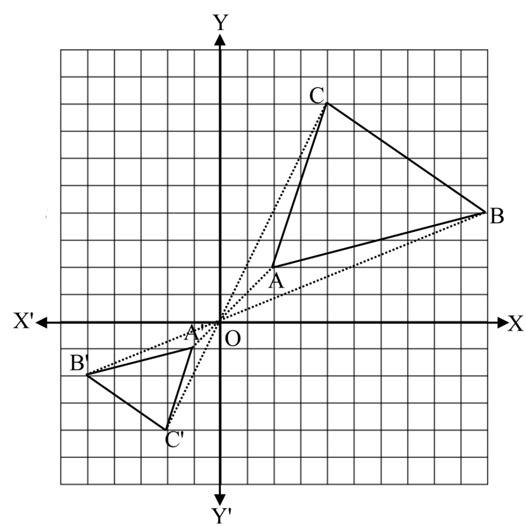
A(1, 1) → A'(2, 2) = A'(2 × 1, 2 × 1)

B(5, 2) → B'(10, 4) = B'(2 × 5, 2 × 2)

C(2, 4) → C'(4, 8) = C'(2 × 2, 2 × 4)

Here, The x and y coordinates of image point are two times the x and y coordinates of corresponding object point (scale factor = k). Based on this, when the center of enlargement is the origin O and the scale factor is k, the image of a point P(x, y) after enlargement is P' (kx, ky).

That is, $P(x, y) \xrightarrow{E[O,\ k]} P'(kx, ky)$



From the given graph, write the coordinates of the vertices of ΔABC and coordinates of vertices of image ΔA'B'C'. Discuss the relationships between the coordinates of vertices of ΔABC and coordinates of vertices of image ΔA'B'C' with your classmates.

The coordinates of the vertices of ΔABC are: A(2, 2), B(10, 4), C(4, 8).

Similarly, the coordinates of the vertices of its image ΔA'B'C' are A' (–1, –1), B' (–5, –2), C' (–2, –4).

Again,

$A(2, 2) \rightarrow A'(-1, -1) = A'(-\frac{1}{2} \times 2, -\frac{1}{2} \times 2)$

$B(10, 4) \rightarrow B'(-5, -2) = B'(-\frac{1}{2} \times 10, -\frac{1}{2} \times 4)$

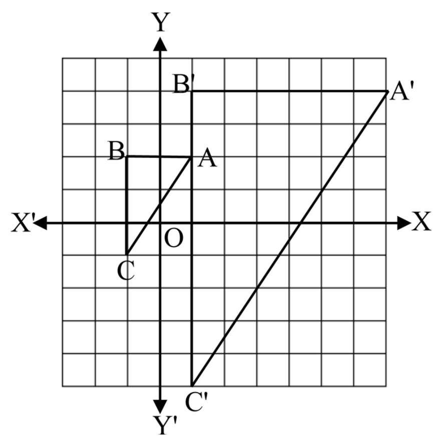
$C(4, 8) \rightarrow C'(-2, -4) = C'(-\frac{1}{2} \times 4, -\frac{1}{2} \times 8)$

Here, The x and y coordinates of image point are $-\frac{1}{2}$ times the x and y coordinates of corresponding object point (scale factor = k). Based on this, when the center of enlargement is the origin O and the scale factor is k, the image of a point P(x,y) after enlargement is P' (kx, ky).

That is, $P(x, y) \xrightarrow{\text{"Enlargement" } E[O, k]} P'(kx, ky)$

b. Trnsformation about the centre of enlargement at any point (a, b)

From the given graph, write the coordinates of the vertices of ΔABC and its image ΔA'B'C'.



Discuss the relationships between the coordinates of ΔABC and its image ΔA'B'C' with your classmates.

The coordinates of the vertices of ΔABC are A(1, 2), B(–1, 2) and C(–1, –1). Similarly, the coordinates of the vertices of its image ΔA'B'C' are A'(7, 4), B'(1, 4) and C'(1, –5)

Again,

$A(1, 2) \rightarrow A'(7, 4) = A'[3\{1 - (-2)\} + (-2), 3(2 - 1) + 1]$

$B(-1, 2) \rightarrow B'(1, 4) = B'[3\{-1 - (-2)\} + (-2), 3(2 - 1) + 1]$

$C(-1, -1) \rightarrow C'(1, -5) = C'[3\{-1 - (-2)\} + (-2), 3(-1 - 1) + 1]$

Based on this, when the center of enlargement is (a, b) and the scale factor is k, the image of a point P(x, y) is P' [$k(x - a) + a$, $k(y - b) + b$].

$\therefore P(x, y) \xrightarrow{E[(a, b), k]} P'[k(x - a) + a, k(y - b) + b]$

Alterntive method

Let A(x, y) be enlarged with scale factor k and center P(a, b).

Then its image will be A'(x', y').

Here, $\overrightarrow{OA} = (x, y)$

$\overrightarrow{OA'} = (x', y')$

$\overrightarrow{OP} = (a, b)$

Now, $\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = (x, y) - (a, b) = (x - a, y - b)$

and $\overrightarrow{PA'} = \overrightarrow{OA'} - \overrightarrow{OP} = (x', y') - (a, b) = (x' - a, y' - b)$

Thus, $\overrightarrow{PA'} = k\,\overrightarrow{PA}$

or, $(x' - a, y' - b) = k(x - a, y - b)$

or, $x' - a = k(x - a)$ ₹ $y' - b = k(y - b)$

$\therefore x' = k(x - a) + a$ $\therefore y' = k(y - b) + b$

Thus, $A(x, y) \xrightarrow{E[P(a,b),k]} A'[k(x - a) + a, k(y - b) + b]$

When the center of enlargement is the origin O and the scale factor is k, then,

$$P(x, y) \xrightarrow{E[O,\, k]} P'(kx, ky)$$

When the center of enlargement is (a, b) and the scale factor is k, then,

$$P(x, y) \xrightarrow{E[(a,\, b),\, k]} P'[k(x - a) + a, k(y - b) + b]$$

Example 1

Enlarge the points A(–3, 4) and B(5, 8) under the following conditions:

a. E [O, 3] b. $E[O, \frac{1}{2}]$ c. E[(1, 2), 2]

Solution: Here,

Given points are A(–3, 4) and B(5, 8)

We Know that,

$$P(x, y) \xrightarrow{E[O,\, k]} P'(kx, ky)$$

a. E[O,3]: Center O(0,0), scale factor $k = 3$

$A(-3, 4) \xrightarrow{E[O,\, 3])} A'[3 \times (-3), 3 \times 4] = A'(-9, 12)$

$B(5, 8) \xrightarrow{E[O,\, 3]} B'(3 \times 5, 3 \times 8) = B'(15, 24)$

b. $E[O, \frac{1}{2}]$: Center O(0, 0), scale factor $k = \frac{1}{2}$. Then,

$A(-3, 4) \xrightarrow{E[O,\, 1\frac{1}{2}]} A'[\frac{1}{2} \times (-3), \frac{1}{2} \times 4] = A'(\frac{-3}{2}, 2)$

$B(5, 8) \xrightarrow{E[O,\, \frac{1}{2}]} B'(\frac{1}{2} \times 5, \frac{1}{2} \times 8) = B'(\frac{5}{2}, 4)$

c. When Centre and scale factor of enlargement are E[(1, 2), 2],

We know that, $P(x, y) \xrightarrow{E[(a,\, b),\, k]} P'[k(x - a) + a, k(y - b) + b]$

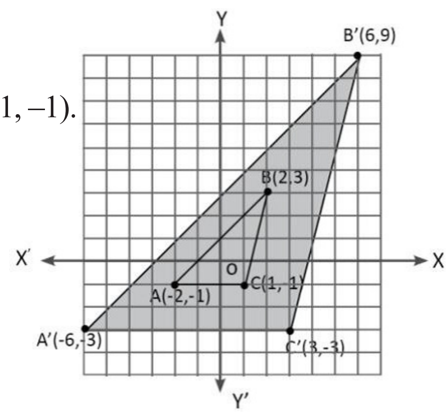
Then, $A(-3, 4) \xrightarrow{E[(1,\, 2),\, 2]} A'[2(-3 - 1) + 1, 2(4 - 2) + 2] = A'(-7, 6)$

$B(5, 8) \xrightarrow{E[(1,\, 2),\, 2]} B'[2(5 - 1) + 1, 2(8 - 2) + 2] = B'(9, 14)$

Example 2

A(–2, –1), B(2, 3) and C(1, 1) are the vertices of triangle ΔABC.

Taking the center of enlargement O(0, 0) and the scale factor $k = 3$, find the coordinates of the vertices of the enlarged image of ΔABC and draw both triangles on the same graph.



Solution: Here,

The vertices of ΔABC are A(–2, –1), B(2, 3), C(1, –1).

The center of enlargement is O(0, 0) and the scale factor is $k = 3$.

We know that: $P(x, y) \xrightarrow{E[O,\, k]} P'(kx, ky)$

Therefore,

$A(-2, -1) \xrightarrow{E[O,\, 3]} A'[3 \times (-2), 3 \times (-1)] = A'(-6, -3)$

$B(2, 3) \xrightarrow{E[O,\, 3]} B'[3 \times 2, 3 \times 3] = B'(6, 9)$

$C(1, -1) \xrightarrow{E[O,\, 3]} C'[3 \times 1, 3 \times (-1)] = C'(3, -3)$

Hence, the coordinates of the image ΔA'B'C' are: A'(–6, –3), B'(6, 9) and C'(3, –3)

Both triangles ΔABC and ΔA'B'C' are shown on the graph.

Example 3

The vertices of ΔPQR are P(3, 0), Q(0, 2) and R (3, 2) by taking the center of enlargement as (1, 1) and the scale factor $k = -2$, find the coordinates of the image ΔP'Q'R'.

Solution: Here,

P (3, 0), Q (0, 2) and R (3, 2) are the vertices of ΔPQR. The center of enlargement is (a, b) = (1, 1) and the scale factor is k = −2.

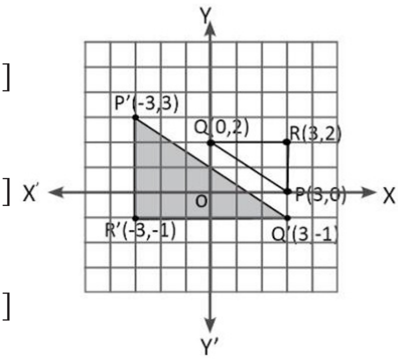
We know that, $P(x, y) \xrightarrow{E[(a, b), k])} P'[k(x - a) + a, k(y - b) + b]$

So,

$P(3, 0) \xrightarrow{E[(1, 1), -2]} P'[-2(3 - 1) +1, -2(0 - 1) +1]$
$= P'(-3, 3)$

$Q(0, 2) \xrightarrow{E[(1, 1), -2]} Q'[-2(0 - 1) +1, -2(2 - 1) +1]$
$= Q'(3, -1)$

$R(3, 2) \xrightarrow{E[(1, 1), -2]} R'[-2(3 - 1) +1, -2(2 - 1) +1]$
$= R'(-3, -1)$



Thus, P'(–3, 3), Q'(3, –1), and R'(–3, –1) are the coordinates of the image triangle ΔP'Q'R'.

The triangles ΔPQR and its image ΔP'Q'R' are shown in the given diagram.

Example 4

An enlargement maps point A(2, 3) to A'(6, 9) and point B(1, 4) to B'(3, 12). Find the center of enlargement and the scale factor.

Solution: Here,

Let the center of enlargement be (a, b) and the scale factor be k.

According to the enlargement formula, if the center is (a, b) and the scale factor is k, then:

$A(2, 3) \xrightarrow{E[(a, b), k]} A'[k(2 - a) + a, k(3 - b) + b] = A'(6, 9)$

So, $k(2 - a) + a = 6$ … (i)

$k(3 - b) + b = 9$ … (ii)

and $B(1, 4)$ $E[(a, b), k])$ $B'[k(1 - a) + a, k(4 - b) + b] = B'(3, 12)$

So, $k(1 - a) + a = 3$ … (iii)

$k(4 - b) + b = 1$ … (iv)

Subtracting equation (iii) from (i)

$k(2 - a) + a = 6$

$k(1 - a) + a = 3$

$k(2 - a - 1 + a) = 3$

Or, $k = 3$

Putting value of k in equation (i)

$3(2 - a) + a = 6$

Or, $6 - 3a + a = 6$

Or, $-2a = 0$

$\therefore a = 0$

Again, putting value of k in equation (ii)

$3(3 - b) + b = 9$

Or, $9 - 3b + b = 9$

Or, $-2b = 0$

$\therefore b = 0$

Thus, the center of enlargement is $(a, b) = (0, 0)$ and the scale factor is $k = 3$.

Alternative method

A(2, 3) →A'(6, 9) = A'(3 × 2, 3 × 3)

B(1, 4) → B'(3, 12) = B'(3 × 1, 3 × 4)

Therefore, comparing with $P(x, y) \xrightarrow{E[O, k]} P'(kx, ky)$

$\therefore$ The center of enlargement is (0, 0)

and the scale factor $k = 3$.

Note: This method can be used only when the center of enlargement is at origin (0, 0).

Exercise 3.2 (C)

1. What is enlargement? Write the definition along with practical examples.
2. In which situations do the figure and its image lie on the same side of the center of enlargement, and in which situations do they lie on opposite sides? Write it.
3. For the enlargement E, find the coordinates of the enlarged images of the points A(4, 5), B(3, 0), C(–2, 3), D(–5, 0), and E(–3, –2) under the given conditions.

   a. E[O, 2] b. E[O, –3] c. $E[O, \frac{3}{2}]$ d. $E[O, \frac{-1}{3}]$

   e. E[(3, –2), 2] f. E[(1, 0), –4] g. $E[(-2, -2), \frac{-3}{2}]$
4. The vertices of ΔABC are A(4, –2), B(3, 1), and C(2, 5). Find the coordinates of the image ΔA'B'C' formed by the enlargement E[O, 2]. Also, present the object and its image on a graph.
5. P(0, –1), Q(1, 3), R(2, 2), and S(1, –2) are the vertices of parallelogram PQRS. Find the coordinates of the image formed by the enlargement E[(1, 3), –2]. Also, present the object and its image on a graph.

6. If an enlargement maps point A to A' and B to B', find the center of enlargement and the scale factor.

a. A(3, 2) → A'(–6, –4)
   B(–2, 4) → B' (4, –8)

b. A(3, –2) → A' (9, –6)
   B(–1, 0) → B' (–3, 0)

c. A(–4, 0) → A' (–10, –1)
   B(4, 6) → B'(6, 11)

d. A(2, 0) → A'(3, –2)
   B(3, 4) → B' (5, 6)

7. If the enlargement E[O, 3] maps point A(a, 2) → A'(9, b), find the values of a and b.

8. If the enlargement E[(a, b), $\frac{1}{2}$] maps point A(–1, 6) → A'(1, 2), find the values of a and b.

9. If a point P(x, y) is reduced by E[O, k_1] and its image is then reduced by E[O, k_2], compare the resulting image with the image obtained by enlarging P(x, y) directly by E[O, $k_1.k_2$]. What conclusion can you draw? Clarify.

10. The vertices of ΔABC are A(1, 3), B(1, 5), C(2, 5).

a. Find the coordinates of ΔA'B'C', the image of ΔABC reflected over the line y = 0.

b. Find the coordinates of ΔA"B"C", the image obtained by enlarging ΔA'B'C' with center (0, 0) and scale factor 2.

c. Draw all three triangles on the same graph.

11. The vertices of ΔABC are A(3, 0), B(0, 2), C(3, 2).

a. Find the coordinates of ΔA'B'C', the image of ΔABC reflected over the line x = 0.

b. Find the coordinates of ΔA"B"C", the image obtained by enlarging ΔA'B'C' with center (0,0) and scale factor 2.

c. Draw all three triangles on the same graph.

12. When ΔABC is enlarged with center (4, 1) and scale factor 2, it gives the image ΔA'B'C' with vertices A'(0, 1), B'(0, 5), C'(6, 5), find the coordinates of the vertices A, B and C.

Answer

1 - 2. Show to the teacher. 3. a. A' (8, 10), B' (6, 0), C'(–4, 6), D' (–10, 0) E' (–6, –4)

b. A'(–12, –15), B'(–9, 0), C'(6, –9), D'(15, 0) E'(9, 6)

c. A'$\left(6, \frac{15}{2}\right)$, B'$\left(\frac{9}{2}, 0\right)$, C'$\left(-3, \frac{9}{2}\right)$, D'$\left(\frac{-15}{2}, 0\right)$ E'$\left(\frac{-9}{2}, -3\right)$

d. A'$\left(\frac{-4}{3}, \frac{-5}{3}\right)$ B'(–1, 0), C'$\left(\frac{2}{3}, -1\right)$, D'$\left(\frac{5}{3}, 0\right)$, E'$\left(1, \frac{2}{3}\right)$

e. A' (5, 12), B'(3, 2) C' (–7, 8), D'(–13, 2), E'(–9, –2)

f. A' (–9, –20), B'(–7, 0), C'(13, –12), D'(25, 0), E'(17, 8)

g. A'$\left(-11, \frac{-25}{2}\right)$, B'$\left(\frac{-18}{2}, -5\right)$, C'$\left(-2, \frac{18}{2}\right)$, D'$\left(\frac{5}{2}, 5\right)$, E'$\left(\frac{-1}{2}, -2\right)$

4. A' (8, –4), B'(6, 2), C' (4, 10) 5. P' (3, 11), Q' (1, 3), R (–1, 5), S'(1, 13)

6. a. E[O, –2] b. E [O, 3] c. E[(2, 1), 2] d. E[(1, 2), 2]

7. a = 3, b = 6 8. a = 3, b = –2 9. Show to the teacher.

10. a. A'(1, –3), B'(1, –5), C'(2, –5) b. A''(2, –6), B''(2, –10), C''(4, –10)

11. a. A'(–3, 0), B'(0, 2), C'(–3, 2) b. A''(–6, 0), B''(0, 4), C''(–6, 4) 12. A(2, 1), B(2, 3), C(5, 3)

Sample Project Work

Collaborative Project Work

Title: Use of Reflection and Rotation

Problem: In a group, perform the following transformations of a triangle with coordinates:

1. Reflect the triangle on the line y = x.
2. Rotate the triangle 90° or 180° around the origin O on both negative and positive directions.
3. Rotate the triangle 90° or 180° around a point (a, b) on both negative and positive directions.

Analyze the nature and shape of the resulting images. Prepare a report with your study findings in the given format and present it for classroom discussion.

Materials Required

1. Photocopy paper 2. List of formulas 3. Graph paper

Student Groups for Project Work:

1. Name and role: 2. Name and role:

3. Name and role: 4. Name and role:

Procedure for Completing the Project Work

All group members should write the formulae for reflection and rotation. Then, using the necessary formulae, transform the triangle according to the given situations above. Find the answer of the following questions:

1. Which formula is used to reflect on the line $y = x$? Confirm it.
2. Which formula is used to rotate 90° or 180° about the origin O in both negative and positive directions? Confirm it.
3. Which formula is used to rotate 90° or 180° about a point (a, b) in both negative and positive directions? Confirm it.
4. When performing reflection and rotation, what is the nature and shape of the figure? Write your observations.

Exploration of Applications and Importance: Investigate the applications of rotation and reflection in different fields and how they can be used. Analyze the effect of rotation around the origin O and about a point (a, b). Study, analyze, and explain your findings.

Conclusion : Study the nature and shape of the object and its image after reflection and rotation on the graph and write your conclusions.

Reflection : How reliable is your study method and conclusion? Did you draw the correct conclusion? What difficulties did you face? Which method was the correct way to reach to your conclusion? Could you have reached the same conclusion using other methods? Were graphs or other methods helpful in reaching your conclusion? How could this be applied in daily life? Reflect on these points.

Points to Include in the Project Report

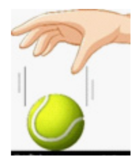
1. Title of the project work
2. Background: Subject matter and significance of the study
3. Objectives: What you aim to find out
4. Study Procedure:
   a. Studied reference materials
   b. Data collection method
   c. Analysis method
   d. Results and presentation
   e. Conclusion
   f. Reflection
   g. Index

Lesson 4 Vector

4.1 Introduction

Vector is an important field of mathematics and science used to solve various problems of daily life based on the direction and magnitude of physical quantities. The concept of vectors was first used by the Irish mathematician William Rowan Hamilton. His contribution to vectors was significant and multifaceted. He introduced the main concepts, terminology and geometric interpretations that laid the foundation for the development of vectors.

Observe the pictures below and answer.

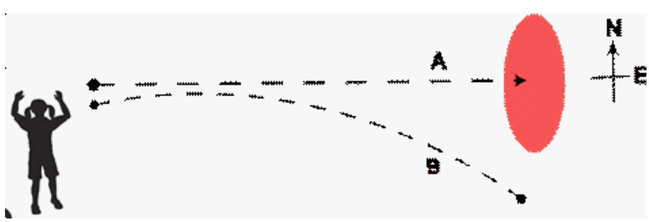
Mass of pumpkin	Speed of a girl going east	Morning time	Acceleration of a falling ball	Height of a child
				

a. Which physical quantities have only magnitud, but no direction?

b) Which physical quantity has both; magnitude and direction?

Discuss your answer with a friend and teacher.

In the picture below, Rima repeatedly tried to throw a ball at the circular mark on the wall. She could not hit the mark many times, but once when she threw ball using slingshot, it went straight at a speed of 15m/s and hit the circular mark.



a. Through path A, the ball reached the mark in a straight line with a speed of 15 m/s.

b. Through path B, the ball did not reach the mark because it moved in a curved path at a speed of 15 m/s.

Path A has a fixed direction towards the east, whereas in path B, the direction of the ball changes continuously while moving. In path A, both the direction and distance are fixed, but in path B, the distance is fixed while the direction keeps changing.

A physical quantity like in picture A, which has both definite magnitude and definite direction, is called a vector. A physical quantity that has magnitude but no definite direction, or does not have direction at all, is called a scalar.

Among weight, force, distance, knowledge, density, area, and emotion; which of these can be measured?

Discuss with your friends and present in the classroom.

Quantities that can be measured are called physical quantities. They are of two types: i. Scalar and ii. Vector

i. Scalar

At what temperature does water boil in general? Discuss.

Water boils at 100°C. Here, the magnitude (100) and unit (°C) fully describe the temperature. Therefore, temperature is a scalar quantity.

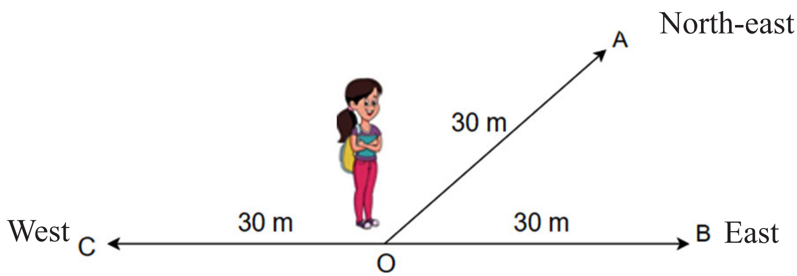
Physical quantities that have only magnitude are called scalar quantities.

Examples: distance, speed, time, temperature, work, power, energy, etc. are scalar quantities.

ii. Vector

Study the figure and answer:

In the figure, if a girl walks 30 m in a straight line from point O, where will she reach?



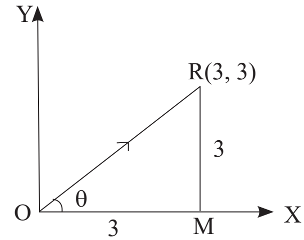
While discussing the above question, if she walks 30 m east, she reaches point B. If she walks west, she reaches point C and if she walks northeast, she reaches point A. In each case, the girl moves from point O toward a specific direction and reaches a definite point. This is an example of displacement.

Therefore, the displacement has a magnitude of 30 m but the direction differs according to the point reached.

Therefore, displacement has both magnitude and direction, so it is a vector quantity.

Physical quantities that have both magnitude and direction are called vector quantities.
Examples: displacement, velocity, acceleration, force, weight, etc.

Types of Vector



In the given figure, the displacement from O to R is represented by $(\overrightarrow{OR})$.
In how many ways can the x- component and y- component of $\overrightarrow{OR}$ be written?

i. Column vector

In the above figure, if $\overrightarrow{OR}$ is represented by $\vec{a}$, since $\vec{a}$ has x- component 3 and y- component 3, it can be written as: $\vec{a} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. When the components of a vector are written vertically and enclosed in small brackets, such vectors are called column vectors.

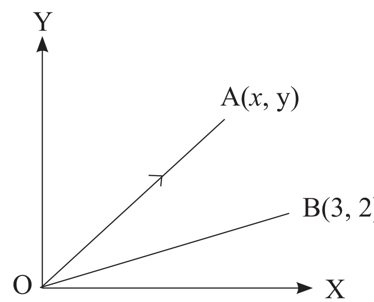
For example: $\vec{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is also a column vector.

ii. Row vector

The above vector $\vec{a} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ can also be written as: $\vec{a} = (3, 3)$. Here, the x- component and y- component are arranged in a row, separated by a comma and enclosed in small brackets. Such vectors are called row vectors.

For example: $\vec{b} = (3, -2)$.

iii. Position vector



In the adjacent figure, what is the initial point of $\overrightarrow{OA}$?
Does $\overrightarrow{OA}$ represent the position of point A? Discuss.

If $\overrightarrow{OA}$ displaces O(0, 0) to A(x, y), then $\overrightarrow{OA}$ represents the position of A(x, y).

Here, $\overrightarrow{OA}$ is called a position vector. The initial point of a position vector is always the origin O(0, 0).

For example: The position vector of B(3, 2) is $\overrightarrow{OB}$, where $\overrightarrow{OB} = (3, 2)$.
Thus, a vector starting from the origin is called the position vector of its final point.

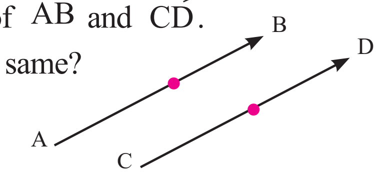
Hence, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of points A and B respectively.

iv. Zero or null vector

Suppose two people are pulling the same rope from its two ends with equal force in opposite directions. In this situation, the net force on the rope is a zero vector, because the two equal forces act in opposite directions balancing each other and resulting in no effective force on either side.
If a vector displaces A(x, y) to A(x, y) itself, what will be its magnitude? Discuss.
Here, the magnitude of the applied vector is zero (0). Such a vector with zero magnitude is called a zero vector.

$\vec{a} = (0, 0)$ is a zero vector.

v. Equal vectors



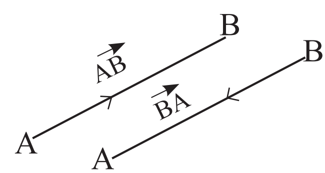
In the adjacent figure, find the magnitude and direction of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Are their magnitudes equal? Are their directions also the same?

Discuss in groups and present your conclusion in the classroom.

Here, $\overrightarrow{AB}$ displaces point A to B with a certain magnitude and direction, and $\overrightarrow{CD}$ displaces point C to D with the same magnitude and in the same direction. Vectors that have equal magnitude and the same direction are called equal vectors.

For example: if $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$ are equal, then it must be $x_1 = x_2$ and $y_1 = y_2$.

vi. Negative vector



In the adjacent figure, find the magnitude and direction of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

Are their magnitudes equal?

Are their directions also the same?

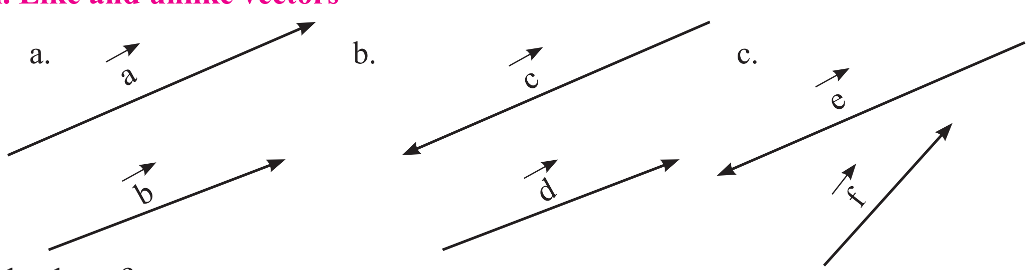
Discuss in groups and present your conclusion in the classroom.

Here, $\overrightarrow{AB}$ displaces point A to B with a certain magnitude and direction, while $\overrightarrow{BA}$ displaces point B to A with the same magnitude but in the opposite direction.

Though their magnitudes (lengths of the line segments) are equal, their directions are opposite.

Vectors having equal magnitudes but opposite directions are called negative vectors. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are negative vectors of each other. Therefore, $\overrightarrow{AB} = -\overrightarrow{BA}$. Example: If $\vec{a} = (6, 2)$ and $\vec{b} = (-6, -2)$, they are negative vectors of each other.

vii. Like and unlike vectors



In the above figure:

i. Which vectors have the same direction?

ii. hich vectors have opposite directions?

iii. Which vectors have neither the same nor opposite directions?

Discuss in groups and present the conclusion in the classroom.

In the figure above, vectors $\vec{a}$ and $\vec{b}$ have the same direction. Thus, vectors having the same direction are called like vectors. For example:

$\vec{a} = (6, 4)$ and $\vec{b} = (3, 2)$ are like vectors because both of them are positive.

But vectors $\vec{c}$ and $\vec{d}$ have opposite directions. Thus, vectors having opposite directions are called unlike vectors. For example:

$\vec{c} = (-6, -4)$ and $\vec{d} = (3, 2)$ are unlike vectors because one vector is negative and the other is positive.

Like or unlike vectors are parallel or collinear vectors.

Similarly, vectors $\vec{d}$ and $\vec{f}$ have neither the same direction nor opposite directions. Therefore, $\vec{e}$ and $\vec{f}$ are neither like or unlike vectors.

4.2 Difference Between Scalar and Vector

Difference between scalar and vector

Scalar	Vector
A physical quantity that has only magnitude is called a scalar.	A physical quantity that has both magnitude and direction is called a vector.
It is expressed by a number and a unit.	It is expressed by a number unit and a direction.
Example: temperature, speed, etc.	Example: velocity, acceleration, etc.

Exercise 4.1

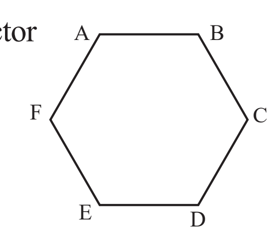
1. Define vector and scalar with examples.
2. Mention differences between scalar and vector quantities.
3. Identify whether the following quantities are vectors or scalars. Write with reasons:

   distance, displacement, force, velocity, speed, work, density, area, volume, acceleration.
4. Define with example.

   a. Row vector b. Column vector c. Position vector

   d. Zero vector e. Equal vectors f. Negative vector

   g. Like vectors h. Unlike vectors
5. In the given regular hexagon ABCDEF, identify equal, like, unlike, and negative vectors.



6. In the adjoining figure, PT||QR, QS||RT and PQ||SR, solve the following questions:

a. Which vector is equal to $\overrightarrow{PQ}$?

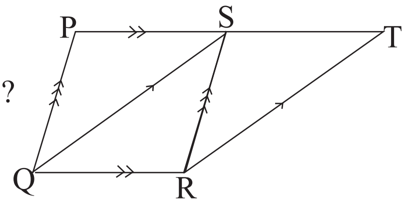
b. Which vector is equal to $\overrightarrow{QS}$?

c. Which two vectors are equal to $\overrightarrow{QR}$?

d. Which vectors are the negative vectors of $\overrightarrow{ST}$?

e. Which is the negative vector of $\overrightarrow{RS}$?

f. Which is the negative vector of $\overrightarrow{TR}$?



Answer

1 - 5. Show to the teacher.

6. a. $\overrightarrow{SR}$ b. $\overrightarrow{RT}$ c. $\overrightarrow{PS}$, $\overrightarrow{ST}$

d. $\overrightarrow{SP}$, $\overrightarrow{RQ}$ e. $\overrightarrow{PQ}$, $\overrightarrow{SR}$ f. $\overrightarrow{RT}$, $\overrightarrow{QS}$

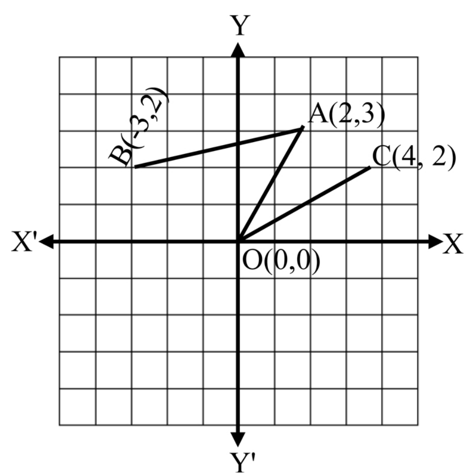
4.3 Representation of Vector in Coordinates and Graph

Plot the following points on the graph:

O(0, 0), A(2, 3), B(–3, 2), C(4, 2) and join the line segments OA, OC and AB.

In the adjoining figure, the line segments OA,OC and AB are shown.

If a displacement occurs from point O(0, 0) to A(2, 3), this displacement is represented by the directed line segment $\overrightarrow{OA}$.



Here, O is the initial point and A is the terminal point after displacement.

Therefore, $\overrightarrow{OA}$ is a vector.

$\overrightarrow{OA}$ represents displacement from O to A, whereas $\overrightarrow{AO}$ represents displacement from A to O.

Thus, $\overrightarrow{OA}$ and $\overrightarrow{AO}$ are different vectors.

Similarly, $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are vectors.

In the figure, if a point moves from O(0, 0) to A(x, y), the displacement is represented by the directed segment $\overrightarrow{OA}$.

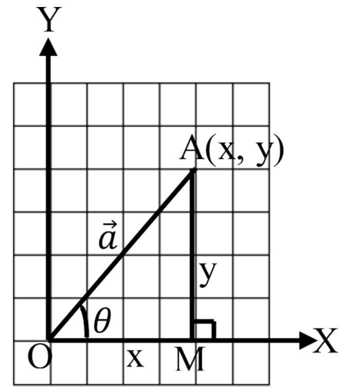
Here, O is the initial point and A is the terminal point after displacement.

Hence, $\overrightarrow{OA}$ is a vector.

$\overrightarrow{OA}$ shows the displacement from O to A.

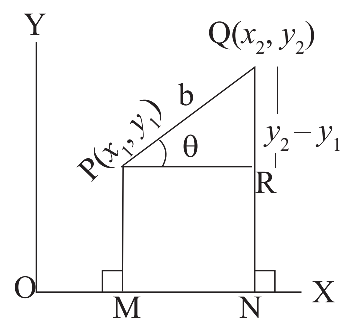
Generally, vectors are denoted by bold face letters in writing. They are shown by placing an arrow over the letter.

In the figure, $\overrightarrow{OA}$ can be denoted by $\vec{a}$ or a. $\vec{a}$ is read as "vector a".



In the diagram, the horizontal displacement of $\overrightarrow{OA}$ is x and the vertical displacement is y. Therefore, in coordinates, vector $\overrightarrow{OA}$ =(x, y) in row and $\begin{pmatrix} x \\ y \end{pmatrix}$ in column form. Here, x is called the x- component and y is called the y- component.

In the figure, when points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are joined, the vector $\overrightarrow{PQ}$ is formed.



From points P and Q, perpendiculars PM and QN are drawn to the X-axis, and from P, PR ⊥ QN is drawn.

Then, $PR = MN = x_2 - x_1$, $QR = y_2 - y_1$

Here, the horizontal displacement (x- component) of $\overrightarrow{PQ}$ $(\vec{b})$, and the vertical displacement (y- component) is QR.

Therefore, in coordinates, $\overrightarrow{PQ}$ is written as (x_2-x_1, y_2-y_1) in row and $\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \end{pmatrix}$ in column form.

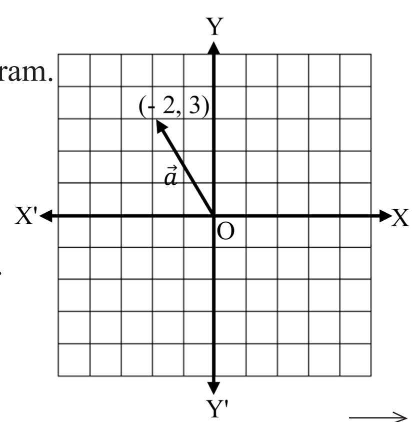
Here, x- component of $\overrightarrow{PQ}$ is $x_2 - x_1$, y- component of $\overrightarrow{PQ}$ is $y_2 - y_1$

Example 1

Represent the vector $\vec{a}$ = (–2, 3) in an arrow diagram.

Solution

While representing $\vec{a}$ =(–2, 3) in an arrow diagram:



Thought Provoking Question: Does a row vector represent only the position?

Example 2

If $\overrightarrow{AB}$ displaces point A(3, 2) to B(6, 5), then draw the diagram representing $\overrightarrow{AB}$ and express $\overrightarrow{AB}$ in the form of column-vector.

Solution

Here, when representing vector $\overrightarrow{AB}$ in the adjacent arrow diagram:

Again, A(3, 2) = (x_1, y_1)

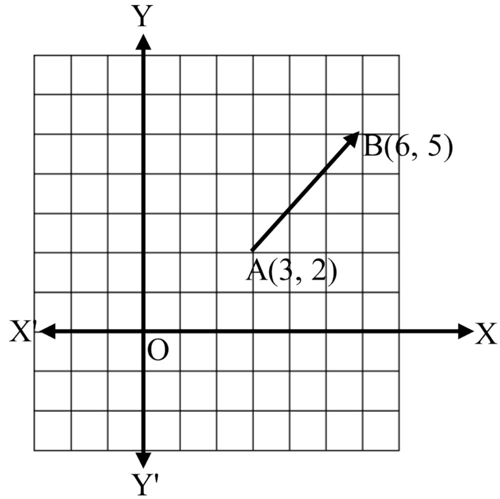
B(6, 5) = (x_2, y_2)

Then, x- component of ($\overrightarrow{AB}$

$= x_2 - x_1 = 6 - 3 = 3$

y- component of $\overrightarrow{AB}$ $= y_2 - y_1 = 5 - 2 = 3$

Therefore, $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$



Exercise 4.2

1. Represent the following vectors in an arrow diagram:

   a. $\vec{a} = (2, 3)$ b. $\vec{b} = (-3, 4)$ c. $\vec{c} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ d. $\vec{d} = (-5, -4)$

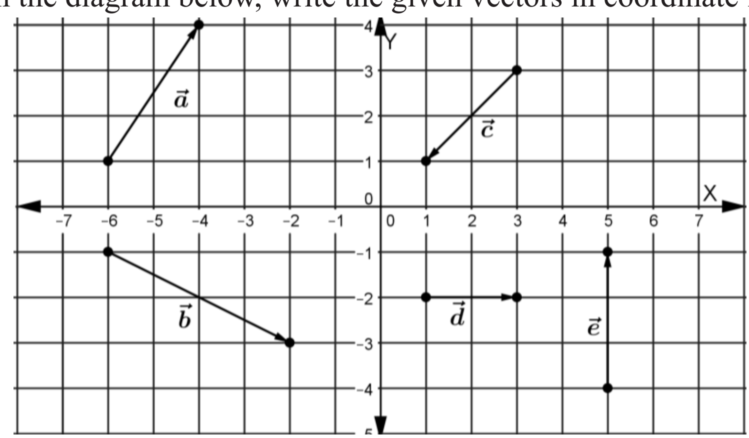
2. Find vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ using the given coordinates. Determine which pairs of vectors are equal:

   a. A(4, 3), B(2, 5) b. A(6, 3), B(5, –4) c. A(–6, 3), B(5, –2)

3. If vector $\overrightarrow{AB}$ displaces point A to B, represent $\overrightarrow{AB}$ in a diagram and express it as a column vector:

   a. A(2, 5), B(–1, 0) b. A(2, 3), B(–5, –4) c. A(–6, 4), B(0, –1)

4. From the diagram below, write the given vectors in coordinate form.



Open Question: Represent the given vectors in a graph using any two points.

a. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ b. $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

Answer

1. Show to the teacher.

2. a. $\overrightarrow{AB} = \begin{pmatrix}-2\\2\end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix}2\\-2\end{pmatrix}$, $\overrightarrow{AB} \neq \overrightarrow{BA}$, b. $\overrightarrow{AB} = \begin{pmatrix}-1\\-7\end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix}1\\7\end{pmatrix}$, $\overrightarrow{AB} \neq \overrightarrow{BA}$

c. $\overrightarrow{AB} = \begin{pmatrix}11\\-5\end{pmatrix}$ $\overrightarrow{BA} = \begin{pmatrix}-11\\5\end{pmatrix}$, $\overrightarrow{AB} \neq \overrightarrow{BA}$

3. a. $\begin{pmatrix}-3\\-5\end{pmatrix}$ b. $\begin{pmatrix}-7\\-7\end{pmatrix}$ c. $\begin{pmatrix}6\\-5\end{pmatrix}$

4. $\vec{a} = \begin{pmatrix}2\\3\end{pmatrix}$, $\vec{b} = \begin{pmatrix}4\\-2\end{pmatrix}$, $\vec{c} = \begin{pmatrix}-2\\-2\end{pmatrix}$, $\vec{d} = \begin{pmatrix}2\\0\end{pmatrix}$, $\vec{e} = \begin{pmatrix}0\\3\end{pmatrix}$

4.4 Triangle Law of Vector Addition

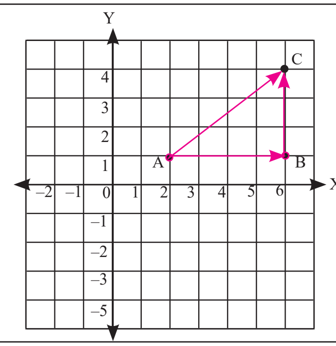
Activity

Problem: Study the graph given on the right and answer the following questions:

a. Write the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in coordinate form.

b. Find the relationship among the vectors $\overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{AC}$.

Procedure: Sit in a suitable group of classmates, discuss the above problems, and draw a conclusion.



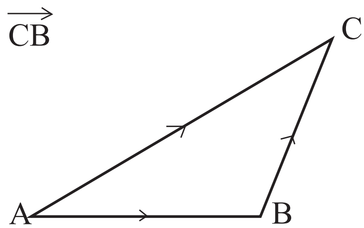
In the diagram on the right:

If $\overrightarrow{AB}$ displaces point A to point B and $\overrightarrow{BC}$ displaces point B to point C, then their combined displacement is $\overrightarrow{AC}$ (displacement from point A to point C).

That is, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{Ac}$

Similarly, $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$ and $\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$

This rule of vector addition is called the triangle law of vector addition.



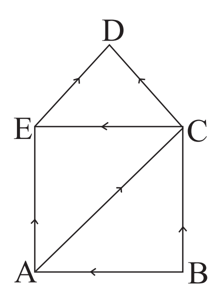
Now, In pantagon ABC

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$ [From triangle law $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CD}$

$= \overrightarrow{AC} + \overrightarrow{CE}$ [From trainalge law $\overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{CE}$

$= \overrightarrow{AE}$ [From Triangle Law]



This is the extended form of the triangle law of vector addition. It is called the polygon law of vector addition.

4.5 Operations of vectors

a. Addition of vectors)

If $\vec{a} = (3, 2), \vec{b} = (6, 4)$ find the value of $\vec{a} + \vec{b}$. Similarly, if $\vec{c} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \vec{d} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ find the value of $\vec{c} + \vec{d}$. Discuss in groups and present your conclusions in the classroom.

Solution

$\vec{a} + \vec{b} = (3, 2) + (6, 4) = (3 + 6, 2 + 4) = (9, 6)$

Similarly, $\vec{c} + \vec{d} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 + 4 \\ 4 + 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$

Thus, when adding vectors, add the x- components together and add the y- components together.

b. Subtraction of vectors

Is it true that $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$? What similarities are there between vector addition and vector subtraction?

Discuss in groups and present the conclusion in the classroom.

Here, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Thus, the subtraction of two positive vectors ($\vec{a}$ and $\vec{b}$) is the same as adding one positive vector ($\vec{a}$) and one negative vector ($-\vec{b}$).

For example If $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ then

$$-\vec{b} = \begin{pmatrix} -x_2 \\ -y_2 \end{pmatrix}$$

Now, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} -x_2 \\ -y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

So, when subtracting vectors, subtract x- component from x- component and y- component from y- component.

4.6 Multiplication of vector by scalar

Given $\vec{a} = (3, 2)$ and $\vec{b} = (6, 4)$. Find their magnitude and direction.

What similarities do they have? Discuss in groups.

Magnitude For $\vec{a}$: $|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$ units

For $\vec{b}$: $|\vec{b}| = \sqrt{6^2 + 4^2} = \sqrt{36+16} = \sqrt{52}$ units

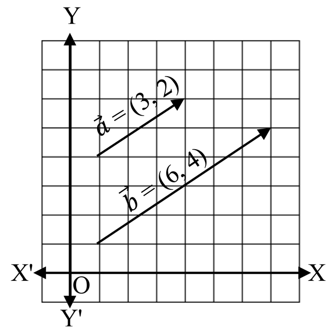
Notice: $|\vec{b}| = 2|\vec{a}|$, which matches our earlier observation.

Direction: The direction (angle with x-axis) of a vector $\vec{v} = (x, y)$ is: $\theta = \tan^{-1}(\frac{y}{x})$

For $\vec{a}$: $\theta_a = \tan^{-1}(\frac{2}{3})$

For $\vec{b}$: $\theta_b = \tan^{-1}(\frac{4}{6}) = \tan^{-1}(\frac{2}{3})$

So, $\vec{a}$ and $\vec{b}$ have the same direction.



Here, when the magnitude of $\vec{a}$ is multiplied by 2, we get $\vec{b}$.
Also, $\vec{a}$ and $\vec{b}$ have the same direction.

Or, $\vec{b} = (6, 4) = 2(3, 2) = 2\vec{a}$

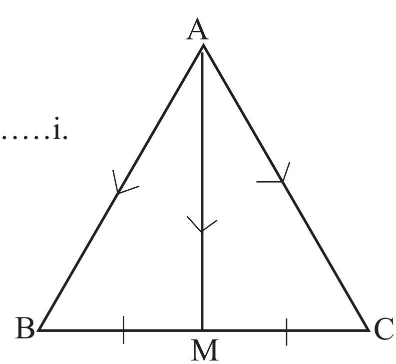
Thus, multiplying vector $\vec{a}$ by a scalar (2) gives vector $\vec{b}$.

> If $\vec{a} = (x, y)$ is a vector and k is any scalar, then multiplying $\vec{a}$ by k gives: $k\vec{a} = k(x, y) = (kx, ky)$, Its magnitude is: $|k\vec{a}| = |k| \cdot |\vec{a}|$. If k is positive, then $\vec{a}$ and $k\vec{a}$ have the same direction. If k is negative, then $\vec{a}$ and $k\vec{a}$ have opposite directions. Whether k is positive or negative, the vectors remain parallel. In the figurem, vectors $\vec{a}$ and $\vec{b}$ have the same direction and $\vec{b} = 2\vec{a}$. Therefore, $\vec{a}$ and $\vec{b}$ are parallel vectors.

Example 1

In ΔABC, if M is the midpoint of BC, then prove that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AM}$

Solution



Here, M is the midpoint of BC in ΔABC

In ΔABM, we can write, $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$ ....i.

In ΔACM and $\overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CM}$ ....(ii)

Adding equations (1) and (2):

$2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} + \overrightarrow{AC} + \overrightarrow{CM}$

$= \overrightarrow{AB} + \overrightarrow{BM} + \overrightarrow{AC} - \overrightarrow{BM}$

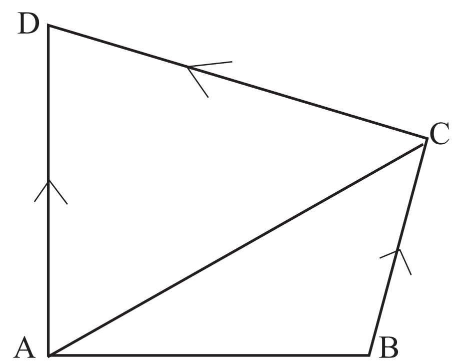
[Since M is the midpoint of BC, $\overrightarrow{CM} = \overrightarrow{MB} = \overrightarrow{BM}$]

$\therefore 2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{AC}$ *proved.*

Example 2

In quadrilateral ABCD, prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 0$

Solution



Here AC is a diagonal of quadrilateral ABCD.

In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ .... (i)

In ΔACD, $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$ ....(ii)

Substitute the value of $\overrightarrow{AC}$ from equation (i) into equation (ii):

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$$

Or, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} - \overrightarrow{AD} = 0$

$\therefore$ $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 0$ *proved.*

Example 3

Prove that $\vec{a} = (1, -2)$ and $\vec{b} = (-3, 6)$ are parallel.

Solution

Here, $\vec{a} = (1, -2)$ and $\vec{b} = (-3, 6)$

Now, $\vec{a} = (-3, 6) = -3(1, -2)$

Or, $\vec{b} = -3\vec{a}$

Hence, $\vec{a}$ and $\vec{b}$ are parallel.

Example 4

If $\vec{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $2\vec{a} - 3\vec{b}$

Solution

Given, $\vec{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ then

Here, $\vec{a} + \vec{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

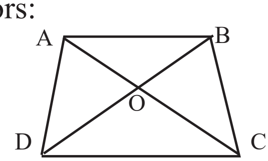
$$\vec{a} - \vec{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$2\vec{a} - 3\vec{b} = 2\begin{pmatrix} 4 \\ 5 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 + 9 \\ 10 - 1 \end{pmatrix} = \begin{pmatrix} 17 \\ 9 \end{pmatrix}$$

Exercise 4.3

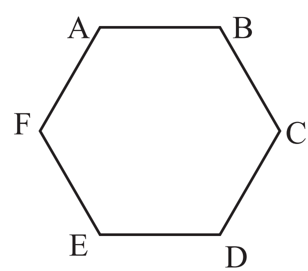
1. If $\overrightarrow{AB} = (-5, 7)$ and $\overrightarrow{BC} = (2, 1)$ find $\overrightarrow{AC}$
2. If $\overrightarrow{OA} = (2, -3)$ and $\overrightarrow{OB} = (0, 2)$ find $\overrightarrow{AB}$
3. If the position vectors of points A and B are $\overrightarrow{OA} = i + 2j$, $\overrightarrow{OB} = 3i - j$, find $\overrightarrow{AB}$
4. If $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$ find $\overrightarrow{OB}$
5. In ΔABC, prove that : $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$
6. In quadrilateral PQRS, prove that: $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{SP} = 0$
7. In a regular pentagon ABCDE, prove that: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$
8. In a parallelogram ABCD, prove that: $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$
9. From the adjoining diagram, find the following vectors:



   a. $\overrightarrow{CA} + \overrightarrow{BC}$ b. $\overrightarrow{DO} + \overrightarrow{OC}$

   c. $\overrightarrow{DC} + \overrightarrow{CA}$ d. $\overrightarrow{DO} + \overrightarrow{AD}$

10. In the regular hexagon, if $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$ express the following vectors in terms of $\vec{a}$ and $\vec{b}$.



   a. $\overrightarrow{AC}$ b. $\overrightarrow{AD}$

   c. $\overrightarrow{AE}$ d. $\overrightarrow{AF}$

11. If $\vec{a} = (4, -2)$, find the following vectors:

   a. $3\vec{a}$ b. $\frac{1}{2}\vec{a}$ c. $\frac{3}{2}\vec{a}$

12. Prove that the following pairs of vectors are parallel in each other.

   a. $\vec{a} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -10 \\ 6 \end{pmatrix}$ b. $\vec{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

13. If $\vec{a} = 3\vec{i} - 2\vec{j}$, $\vec{b} = 6\vec{i} + k\vec{j}$ and $\vec{a} // \vec{b}$ find the value of k.
14. If $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, then find the following vectors:

   a. $\vec{a} + \vec{b}$ b. $\vec{a} - \vec{b}$ c. $\vec{b} - \vec{a}$

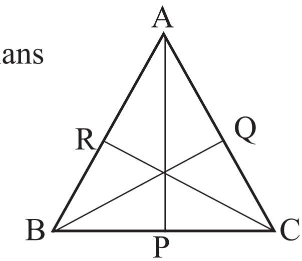
   d. $2\vec{a} + \vec{b}$ e. $3\vec{a} - 2\vec{a}$ f. $\frac{1}{2}\vec{a} + \frac{3}{2}\vec{b}$

15. a. If $\vec{a} - \vec{b} = (12, 4)$ and $\vec{b} = (5, 7)$, then find $\vec{a}$.

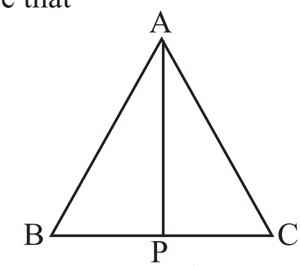
b. If $\vec{a} + \vec{b} = (5, 1)$ and $\vec{a} = (0, 4)$ find $\vec{b}$.

c. If $2\vec{a} + 3\vec{b} = (0, -7)$ and $\vec{b} = (2, -3)$ find $\vec{a}$.

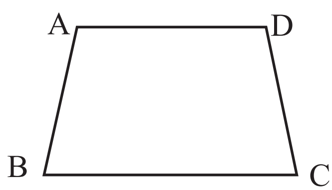
16. In the given triangle ΔABC, AP, BQ and CR are medians
Prove that $\overrightarrow{AP} + \overrightarrow{BQ} + \overrightarrow{CR} = 0$



17. In the triangle ΔABC, if point P lies on side BC, prove that
$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CP} - \overrightarrow{BP}$



18. In the given quadrilateral ABCD, prove that: $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{BC}$



19. If four points are respectively A(1, –2), B(2, –5), C(4, 5) and D(5, 2), prove that $\overrightarrow{AC} = \overrightarrow{BD}$

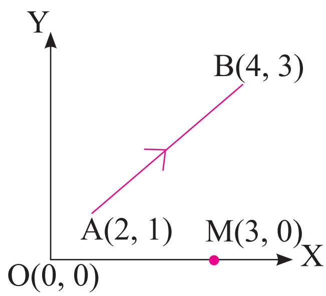
20. If P(–2, 3), Q(–5, 6), R(2, 7) and S(5, 4) are given points, prove that $\overrightarrow{PQ} = -\overrightarrow{RS}$.

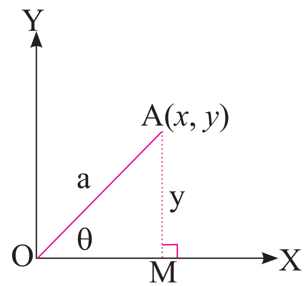
Answer

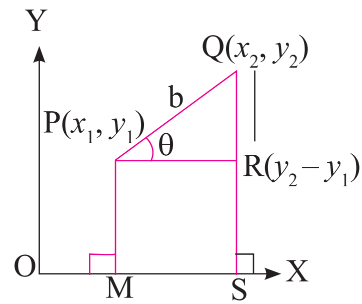
1. (–3, 8)	2. (–2, 5)	3. $2\vec{i} - 3\vec{j}$	4. (11, 11)	
9.a. $\overrightarrow{BA}$	b. $\overrightarrow{DC}$	c. $\overrightarrow{DA}$ d. $\overrightarrow{AO}$	10. a. $\vec{a} + \vec{b}$	
b. $2\vec{b}$	c. $2\vec{b} - \vec{a}$	d. $\vec{b} - \vec{a}$	11. a. (12, –6)	b. (2, –1)
c. (6, –3)	13. –4	14. a. (–2, 6)	b. (4, –2) c. (–4, 2)	
d. (–1, 8)	e. (9, –2)	f. (–4, 7)	15. a. (17, 11)	b. (5, –3) c. (–3, 1)

4.7 Magnitude, Direction of a Vector and Unit Vector

Magnitude of a vector







By studying the figures above, how can we find the lengths of OM, AB, OA and PQ? Discuss in groups and present the conclusion in the classroom.

The lengths of the line segments mentioned above can be found by using the distance formula between any two points. The length of a directed line segment representing a vector is called the magnitude of that vector.

The magnitude of vector $\vec{a}$ is denoted by $|\vec{a}|$ and is read as the absolute value of vector a.

In the figures above:

The length of OM is = $\sqrt{(3-0)^2+(0-0)^2} = \sqrt{(3)^2+(0)^2} = \sqrt{9+0} = \sqrt{9} = 3$ unit

The length of AB is = $\sqrt{(4-2)^2+(3-1)^2} = = \sqrt{(2)^2+(2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

Therefore, the magnitude of vector $\overrightarrow{AB}$ is $|\overrightarrow{AB}| = 2\sqrt{2}$ unit

Similarly, the length of OA is $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{(x)^2+(y)^2} = \sqrt{x^2+y^2}$ unit

Therefore, the magnitude of vector $\overrightarrow{OA}$ is $|\overrightarrow{OA}| = \sqrt{x^2+y^2}$ unit

Likewise, the length of PQ is $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ unit

Therefore, the magnitude of vector $\overrightarrow{PQ}$ is $|\overrightarrow{PQ}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ unit.

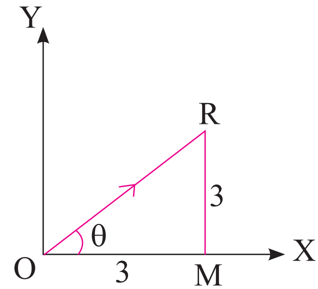
In general, the magnitude of a vector is: $|\vec{a}| = \sqrt{(X-\text{component})^2+(Y-\text{component})^2}$.
For example, if $\vec{a} = (3,4)$,
then $|\vec{a}| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$ units.

Since length is always expressed using a positive sign, the magnitude of a vector is also always positive.

Direction of a vector

In the adjacent figure, what will be the angle made by line segment $\overrightarrow{OR}$ with the positive direction of the X- axis?

Discuss in groups and present the conclusion in the classroom.



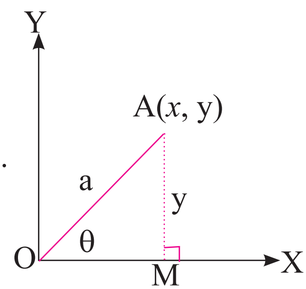
Let the angle made by line segment $\overrightarrow{OR}$ with the positive X- axis be θ.

$\tan\theta = \frac{RM}{OM} = \frac{3}{3} = 1 = \tan 45°$

$\therefore \theta = 45°$

Here, $\overrightarrow{OR}$ makes an angle of 45° with the positive X- axis.

This angle 45° is called the direction of vector $\overrightarrow{OR}$.



Similarly,

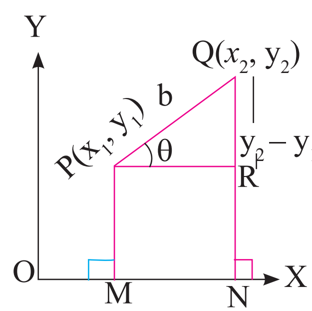
Let the angle made by $\overrightarrow{OA}$ with the positive X- axis be θ.

$\tan\theta = \frac{AM}{OM} = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Thus, the direction of $\overrightarrow{OA}$ is $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ degrees.

Likewise,



In the adjacent figure, let $\overrightarrow{PQ}$ make an angle θ with the positive X- axis:

$$\tan\theta = \frac{QR}{PR} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

Thus, direction of $\overrightarrow{PQ}$ $(\theta) = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

The angle made by any vector with the positive direction of the X– axis is called the direction of that vector $\theta = \tan^{-1}\left(\frac{\text{y- component}}{\text{x- component}}\right)$

Example: For

$\vec{a} = (3, 2)$ direciton of $\vec{a}$ $(\theta) = \tan^{-1}\left(\frac{2}{3}\right) = 33.69°$

Given $\vec{a} = (1, 0)$, $\vec{b} = (0, 1)$ find their magnitudes. Discuss in groups and present the conclusion in the classroom. Are the magnitudes of $\vec{a}$ and $\vec{b}$ equal to 1 unit?

A vector whose magnitude is 1 unit is called a unit vector.

For $\vec{a} = (1, 0)$ the magnitude is $|\vec{a}| = \sqrt{1^2 + 0^2} = \sqrt{1 + 0} = \sqrt{1} = 1$

Similarly, for $\vec{b} = (0, 1)$ the magnitude is $|\vec{b}| = \sqrt{0^2 + 1^2} = \sqrt{0+1} = \sqrt{1} = 1$ unit.

Thus, both $\vec{a}$ and $\vec{b}$ have magnitude 1, so they are unit vectors.

If any vector $\vec{a}$ is divided by its magnitude $|\vec{a}|$, the resulting vector is the unit vector in the direction of $\vec{a}$. It is denoted by $\hat{a}$

i.e, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

For example, if $\hat{a} = (3,4)$, then the unit vector in the direction of $\vec{a}$ is

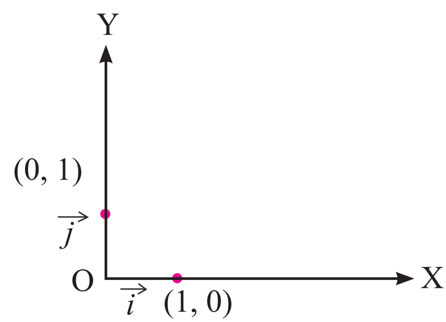
or, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{(3,4)}{\sqrt{3^2+4^2}} = \dfrac{(3,4)}{5} = \left(\dfrac{3}{5}, \dfrac{4}{5}\right)$

Unit vector parallel to X axis and Y axis

The unit vectors parallel to X- axis is denoted by $\vec{i} = (1,0)$ and to Y- axis is denoted by $\vec{j} = (0,1)$

If $\vec{a} = (x, y)$ then it can be written as $\vec{a} = x\vec{i} + y\vec{j}$

$$\begin{aligned}\vec{a} &= x\vec{i} + y\vec{j}\\ &= x(1,0) + y(0,1)\\ &= (x, 0) + (0, y)\\ &= (x + 0, 0 + y)\\ &= (x, y)\end{aligned}$$



Example 1

If $\vec{a} = (-4, 3)$, find its magnitude, direction, and the unit vector in the direction of $\vec{a}$.

Solution

Here, $\vec{a} = (-4, 3)$

X- component $(x) = -4$

Y- component $(y) = 3$

Now, magnitude of $\vec{a} = |\vec{a}| = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ unit

Direction of $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{-4}\right)$

Here, the x- component is negative and y- component is positive. So, it lies in second gudrant.

So, $\theta = 180^0 - 36.87^0$

$\therefore \theta = 143.13^0$

Again

Unit vector of $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(-4,3)}{5} = \left(\frac{-4}{5}, \frac{3}{5}\right)$

Hence, magnitude of $\vec{a} = 5$ units and direction of $\vec{a} = \theta = 143.13^0$ and unit vector of $\vec{a}$ is $= \left(\frac{-4}{5}, \frac{3}{5}\right)$

Example 2

If vector $\overrightarrow{AB}$ displaces point A(5, 3)→B(8, 1) and vector $\overrightarrow{PQ}$ P(2, 0) displaces point P(2, 0)→Q(–1, 2), then prove that $|\overrightarrow{AB}| = |\overrightarrow{PQ}|$

Solution

For $\overrightarrow{AB}$,

Let $A(5, 3) = (x_1, y_1)$ and

$B(8, 1) = (x_2, y_2)$

The magnitude of vector $\overrightarrow{AB}$ is

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(8-5)^2 + (1-3)^2}$$
$$= \sqrt{(3)^2 + (-2)^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13} \text{ unit}$$

For $\overrightarrow{PQ}$

Let $P(2, 0) = (x_1, y_1)$ and $Q(-1, 2) = (x_2, y_2)$

The magnitude of vector $\overrightarrow{PQ}$ is $|\overrightarrow{PQ}|$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1-2)^2 + (2-0)^2}$$
$$= \sqrt{(-3)^2 + (2)^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13} \text{ units}$$

$|\overrightarrow{AB}| = |\overrightarrow{PQ}|$ *proved.*

Example 3

If vector $\overrightarrow{AB}$ displaces point A(3, 1)→B(5, –2), express $\overrightarrow{AB}$ in the form $xi + yj$ and also find the unit vector in the direction of $\overrightarrow{AB}$.

Solution

Here, $A(3, 1) = (x_1, y_1)$ and

$B(5, -2) = (x_2, y_2)$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$
$$= \begin{pmatrix} 5-3 \\ -2-1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= 2\vec{i} - 3\vec{j}$$

The magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

Unit vector in the direction of $\overrightarrow{AB} = \widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{(2,-3)}{\sqrt{13}} = \left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$

Example 4

If $\vec{a} = (x, 1)$ is a unit vector, find the value of x.

Solution

Since $\vec{a} = (x, 1)$ is a unit vector, its magnitude must be 1. $|\vec{a}| = 1$

again, $|\vec{a}| = \sqrt{x^2 + 1^2}$

Or, $1 = \sqrt{x^2 + 1}$

Square both sides: $x^2 + 1 = 1^2$

Or, $x^2 + 1 = 1$

Or, $x^2 = 0$

Thus, $x = 0$

Example 5

If $\vec{a} = (2, 1)$, $\vec{b} = (6, 3)$, prove that $\vec{a}$ and $\vec{b}$ are equal (parallel) vectors.

Solution

Vectors having the same direction are called equal (parallel) vectors. So, to show that $\vec{a}$ and $\vec{b}$ are equal vectors, we must show that they both have the same direction.

Here, $\vec{a} = (2, 1) = (x, y)$

Direction of $\vec{a}$ $(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Similarly, $\vec{b} = (6, 3) = (x, y)$

Direction of $\vec{b}$ $(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{6}\right) = = \tan^{-1}\left(\frac{1}{2}\right)$

Hence we found the direction of $\vec{a}$ and $\vec{b}$ are same.

Example 6

If vector $\overrightarrow{AB}$ displaces $A(2, 2) \rightarrow B(5, 6)$ and vector $\overrightarrow{CD}$ displaces $C(3, 0) \rightarrow D(6, 4)$, prove that: $\overrightarrow{AB} = \overrightarrow{CD}$

Solution

Vectors that have equal magnitude and the same direction are called equal vectors. So, to show $\overrightarrow{AB} = \overrightarrow{CD}$, we must prove that both vectors have the same magnitude and the same direction.

Suppose, $A(2, 2) = (x_1, y_1)$ and

$B(5, 6) = (x_2, y_2)$

Now, $\overrightarrow{AD} = (x_2 - x_1, y_2 - y_1)$

$= (5 - 2, 6 - 2) = (3, 4)$

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ units

Direction of $\overrightarrow{AB}$ $(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right)$

Again

Suppose, $C(3,0) = (x_1,y_1)$ and

$D(6,4) = (x_2,y_2)$

Now, $\overrightarrow{CD} = (x_2 - x_1, y_2 - y_1)$

$= (6 - 3, 4 - 0)$

$= (3, 4)$

Magnitude of $\overrightarrow{CD} = |\overrightarrow{CD}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ units

Magnitude of $\overrightarrow{CD}$ $(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right)$

Therefore, $\overrightarrow{AB} = \overrightarrow{CD}$

Thus, the two vectors are equal vectors.

Example 7

If the points are A(6, 1), B(a, b) and $\overrightarrow{AB} = \begin{pmatrix}-1\\0\end{pmatrix}$, find the values of a and b.

Solution

Suppose, A(6, 1) = (x_1, y_1)

B(a ,b) = (x_2, y_2)

Now, $\overrightarrow{AB} = \begin{pmatrix}x_2 - x_1\\y_2 - y_1\end{pmatrix} = \begin{pmatrix}a-6\\b-1\end{pmatrix}$

According to questions

$\overrightarrow{AB} = \begin{pmatrix}-1\\0\end{pmatrix}$

or, $\begin{pmatrix}a-6\\b-1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

So

$a - 6 = -1 \quad \therefore a = 5 \qquad\qquad b - 1 = 0 \quad \therefore b = 1$

Thus, $a = 5$ and $b = 1$

Exercise 4.4

1. Define with examples:

   a. Unit vector b. Magnitude of a vector c. Direction of a vector

2. Find the magnitude and direction of each vector below:

   a. $\vec{a} = (3, 3)$ b. $\vec{b} = (-4, 3)$ c. $\vec{a} = (-5, 5\sqrt{3})$

3. If $\overrightarrow{PQ}$ displaces point P to point Q, express $\overrightarrow{PQ}$ in column form. Also find its magnitude and direction:

   a. P(2, –2), Q(7, –5) b. P(4, –2), Q(6, 1)

4. If $\overrightarrow{AB}$ displaces point A to B and $\overrightarrow{CD}$ displaces point C to D, prove that: $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

   a. A(–5, 4), B(0, 2), C(1, –1), D(6, –3)

   b. A(4, 5), B(7, –3), C(–1, –3), D(2, –11)

5. Points A(–3, 2), B(2, 4), C(x, 3), D(2, –2) are given. If $|\overrightarrow{AB}| = |\overrightarrow{CD}|$, find the value of x.

6. Find the unit vector in the direction of the given vectors:

   a. $\vec{a} = (-3, 4)$ b. $\vec{b} = (2, -5)$

7. If vector $\overrightarrow{AB}$ displaces point A to B, express $\overrightarrow{AB}$ in the form $xi + yj$ and also find the unit vector in the direction of $\overrightarrow{AB}$:

a. A(5, 6), B(–2, 0)   b. A(–2, 1), B(–1, –2)

8. a. If $\vec{a} = (\frac{3}{5}, y)$ is a unit vector, find the value of y.

b. If $\vec{b} = (\frac{-3}{\sqrt{13}}, \frac{y}{\sqrt{13}})$ is a unit vector, find the value of y.

9. Determine whether the following pairs of vectors are equal or unequal:

a. $\vec{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ **and** $\vec{b} = \begin{pmatrix} -6 \\ -15 \end{pmatrix}$ (b) $\vec{a} = (3, -6)$ **and** $\vec{b} = (1, -2)$

10. a. If $\overrightarrow{AB}$ displaces A(2, 1) → B(3, 3) and $\overrightarrow{CD}$ displaces C(–2, –6) → D(1, 2), prove: $\overrightarrow{AB} = \overrightarrow{CD}$

b. If $\overrightarrow{PQ}$ displaces P(2, –3) → Q(4, –2) and $\overrightarrow{RS}$ displaces R(1, –5) → S(–1, –5), prove: $\overrightarrow{PQ} = \overrightarrow{RS}$

11. a. (a) Let A(0, 3), B(2, 5), C(2, 3) be three points. If $\overrightarrow{AB} = \overrightarrow{CD}$ find the coordinates of D.

b. Let $A(-1, y)$, $B(0, 4)$, $C(-1, 3)$ and $D(x, 6)$ be four points. If $\overrightarrow{AB} = \overrightarrow{CD}$ find the values of x and y.

12. Points A(6,9),B(a,b)are given. If $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ find the values of a and b.

Answer

1. Show to the teacher.

2. (a) $3\sqrt{2}, 45^0$ (b) $5, 143.13^0$ (c) $10, 240^0$

3.(a) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}, \sqrt{34}, 329.03^0$ (b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \sqrt{13}, 56.3^0$

5. $x = 0$ or 4

6.(a) $\left(\frac{-3}{5}, \frac{4}{5}\right)$ (b) $\left(\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}\right)$

7.(a) $-7\vec{\imath} - 6\vec{\jmath}, \left(\frac{-7}{\sqrt{85}}, \frac{-6}{\sqrt{85}}\right)$ (b) $\vec{\imath} - 3\vec{\jmath}, \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$

8.(a) $y = \pm\frac{4}{5}$ (b) $y = \pm 2$

9.(a) unlike   (b) unlike

11.(a) (0, 5)   (b) $x = 0$, y = 1   12. a = 8, b = 10

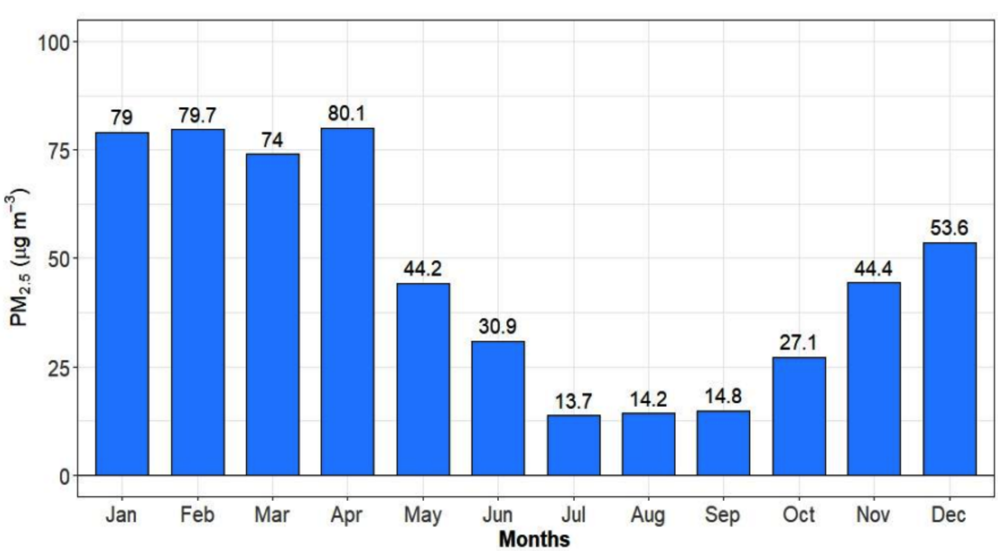
Lesson 5 Statistics

Statistics is a branch of mathematics that deals with the collection, presentation, classification, and analysis of data. The origin of statistics can be traced back to ancient civilizations such as Egypt, Babylon, and Rome, where people used simple methods to keep records of population size, trade and taxes. However, the development of modern statistics is credited to the European Renaissance, during which it evolved through scientific methods, critical thinking, and empirical observations. Prominent statisticians of the 18^{th} centuries, such as Leonhard Euler, Thomas Bayes and Pierre Simon Laplace, made significant contributions to the advancement of statistics.

Since statistics is a branch of mathematical science, it is applied in various fields such as natural sciences, social sciences, humanities, and commerce. Statistics can be used to describe numerical data related to any subject, which is known as descriptive statistics. In addition, it is used to present data in various forms and patterns for better understanding and interpretation.

5.1.1 Quartiles and Percentiles of Individual and Discrete data

The table presents data on the amount of harmful particles smaller than 2.5 micrograms per cubic meter in the air of Lalitpur in 2023, according to the Khumaltar Air Pollution Monitoring Center.



The amount of such fine particles is good for health only up to 20 level and if it exceeds that level, it is harmful.

Study the above picture and answer the following questions:

a. Classify the months into two groups based on good and bad air quality.

b. Find the one year average of the air pollution conditions in Lalitpur.

c. What could be the reasons for more pollution? Identify the month and give your arguments about the appropriate reasons.

Activity 1

Write down the marks obtained by the friends of your class in the first terminal examination and try to answer the following questions.

a. Present the obtained data (marks) in an ascending order.

b. In the data written in an ascending order, whose marks obtained divide the data into four equal parts? Present.

Helpful example for doing the above activity 1: The temperature recorded in Kathmandu every 2 hours on a certain day of winter season is as follows:

5°C, 7°C, 3°C, 10°C, 12°C, 15°C, 9°C, 17°C, 13°C, 19°C, 8°C

a. Present the given data in an ascending order of temperature.

b. How many degrees Celsius divide the data written in an ascending data into two equal parts?

c. Discuss how many (numbers/temperature data. should be taken to divide the given data into four equal parts.

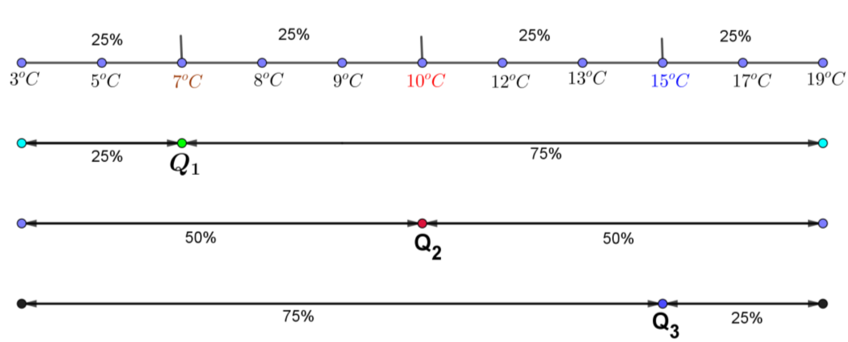
The data is written in an ascending order as follows:

3°C, 5°C, 7°C, 8°C, 9°C, 10°C, 12°C, 13°C, 15°C, 17°C, 19°C

When the given data is written in an ascending order, 10°C lies at the middle of the data and divides the data into two equal parts. Therefore, the 10°C is the median.

Three points (number/temperature) divide the data into four equal parts.

Study the given figure:



Counting from the left, the third item i.e., 7°C, the sixth item i.e., 10°C and the ninth item i.e., 15°C divide the series into four equal parts. The values that divide a data set into four equal parts are called quartiles. Here, 7°C is called the first quartile or lower quartile (Q_1), 10°C is called the second quartile (Q_2) or median (M_d) and 15°C is called the third quartile or upper quartile (Q_3).

The value that divides the given data set into below 25% or above 75% of the data is called the first quartile or lower quartile. The value that divides the given data into two equal parts, the lower quartile 50% and the upper quartile 50%, is called the second quartile or the median. The value that divides the given data set into above 25% and below 75% is called the third quartile or upper quartile

5.1.2 Quartile Values in Individual Data

In the above example, how can we find the quartile values when there are a total of 11 data points in the individual data? Now, if there are N items in an individual data, how can we find the quartile values of the first quartile (Q_1), the second quartile (Q_2) and the third quartile (Q_3)?

Discuss.

Where, the total number of data points = N

First quartile (Q_1) $= \left(\frac{N+1}{4}\right)^{th}$ item

Second quartile (Q_2) $= 2\left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{N+1}{2}\right)^{th}$ item

Third quartile (Q_3) $= 3\left(\frac{N+1}{4}\right)^{th}$ item

Thought Provoking Question: Why is it divided by 4?

5.1.3 Quartile Values in Discrete Data

Similarly, if the sum of frequencies in discrete data is given as Σf, how can the quartile values be found? Discuss.

Where, the sum of frequencies (sum frequency) is $\Sigma f = N$

Likewise, in individual data, the quartile values in discrete data can be found in the same way.

First quartile (Q_1) $=\left(\frac{N+1}{4}\right)^{th}$ item

Second quartile (Q_2) $=2\left(\frac{N+1}{4}\right)^{th}$ item $=\left(\frac{N+1}{2}\right)^{th}$ item

Third quartile (Q_3) $=3\left(\frac{N+1}{4}\right)^{th}$ item

Example 1

Find the three quartile values based on the given weights of 11 students studying in class 9.

35 kg, 39 kg, 52 kg, 25 kg, 32 kg, 28 kg, 46 kg, 41 kg, 42 kg, 38 kg, 50 kg

Solution: Here,

When the given data is written in ascending order,

25 kg, 28 kg, 32 kg, 35 kg, 38 kg, 39 kg, 41 kg, 42 kg, 46 kg, 50 kg, 52 kg

The total number of data (N) = 11

Now according to the formula,

First quartile (Q_1) $=\left(\frac{N+1}{4}\right)^{th}$ item $=\left(\frac{11+1}{4}\right)^{th}$ item $=\left(\frac{12}{4}\right)^{th}$ item $=3^{th}$ item = 32 kg

Second quartile (Q_2) $=2\left(\frac{N+1}{4}\right)^{th}$ item $=(2\times 3)^{th}$ item $=6^{th}$ item = 39 kg $\left(\because \left(\frac{N+1}{4}\right)=3\right)$

Third quartile (Q_3) $=3\left(\frac{N+1}{4}\right)^{th}$ item $=3\times 3^{th}$ item $=9^{th}$ item = 46 kg

So, first quartile (Q_1) = 32 kg, second quartile (Q_2) = 39 kg and third quartile (Q_3) = 46 kg

Example 2

A rural municipality has defined the type of workers based on experience and has arranged for them to get the daily wages as given below.

Rs. 1100, Rs. 750 kg, Rs. 900, Rs. 1200, Rs. 950, Rs. 1000, Rs. 1150, Rs. 800

Based on this, answer the questions given below:

a. Find the first quartile (Q_1), second quartile (Q_2) and third quartile (Q_3).

b. Describe the relationship between the second quartile (Q_2) and median (M_d).

Solution: Here,

When the given data is written in ascending order,

Rs. 750 kg, Rs. 800, Rs. 900, Rs. 950, Rs. 1000, Rs. 1100, Rs. 1150, Rs. 1200

The total number of data (N) = 8

Now, according to the formula:

a. First quartile (Q_1)

$$(Q_1) = \left(\frac{N+1}{4}\right)^{th} \text{ item} = \left(\frac{8+1}{4}\right)^{th} \text{ item} = \left(\frac{9}{4}\right)^{th} \text{ item} = 2.25^{th} \text{ item}$$

$= 2^{th}$ item + 0.25 (3^{rd} item – 2^{nd} item)

= Rs. 800 + 0.25 (Rs. 900 – Rs. 800)

= Rs. 800 + 0.25 (Rs. 100)

= Rs. 800 + Rs. 25

= Rs. 825

Second quartile (Q_2) $= 2\left(\frac{N+1}{4}\right)^{th}$ item $= 2 \times 2.25^{th}$ item $\because \left(\frac{N+1}{4}\right)^{th}$ item $= 2.25^{th}$ item

$= 4.5^{th}$ item

$= 4^{th}$ item + 0.5 (5^{th} item – 4^{th} item)

= Rs. 950 + 0.5 (Rs. 1000 – Rs. 950)

= Rs. 950 + 0.5 (Rs. 50)

= Rs. 950 + Rs. 25

= Rs. 975

Third quartile (Q_3) $= 3\left(\frac{N+1}{4}\right)^{th}$ item $= 3 \times 2.25^{th}$ item $\because \left(\left(\frac{N+1}{4}\right)^{th}$ item $= 2.25^{th}$ item$\right)$

$= 6.75^{th}$ item

$= 6^{th}$ item + 0.75 (7^{th} item – 6^{th} item)

= Rs. 1100 + 0.75 (Rs. 1150 – Rs. 1100)

= Rs. 1100 + 0.75 (Rs. 50)

= Rs. 1100 + Rs. 37.5

= Rs. 1137.5

Therefore, First quartile (Q_1) = Rs. 825, second quartile (Q_2) = Rs. 975
Third quartile (Q_3) = Rs. 1137.5

b. Relation between median and second quartile from above,

Second quartile (Q_2) $= 2\left(\frac{N+1}{4}\right)^{th}$ item $= 4.5^{th}$ item = Rs. 975

Median (M_d) $= \left(\frac{N+1}{2}\right)^{th}$ item $= 4.5^{th}$ item = Rs. 975

Hence, the value of the second quartile (Q_2) is equal to the median (Md).

Therefore, $Q_2 = M_d$.

Example 3

In a survey conducted by a municipality, the following data show the ages of teachers and the number of teachers teaching at each age. Based on this data, answer the following questions:

Age (years)	25	30	35	45	50
Number of teachers	5	8	10	7	1

a. Find the first quartile (Q_1) b. Find the third quartile (Q_3)

Solution: Here,

Let us present the given data in a cumulative frequency table:

Age (years)	No. of teachers (*f*)	Cumulative frequency (*cf*)
25	5	5
30	8	5 + 8 = 13
35	10	13 + 10 = 23
45	7	23 + 7 = 30
50	1	30 + 1 = 31
	$\sum f = N = 31$	

From the data given, the total frequencies ($\sum f$) = N = 31

According to the formula:

a. The first quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{31+1}{4}\right)^{th}$ item $= \left(\frac{32}{4}\right)^{th}$ item = 8^{th} item

From the cumulative frequency table, the value of 8^{th} item = 30 years.

Hence, first quartile (Q_1) =30 years

b. Similarly, the third quartile $(Q_3) = 3\left(\frac{N+1}{4}\right)^{th}$ item

$= 3 \times 8^{th}$ item $= 24^{th}$ item [because $\left(\frac{N+1}{4}\right)^{th}$ item $= 8^{th}$ item]

From the cumulative frequency table, the value of the 24^{th} item = 45.

Hence, third quartile (Q_3) = 45 years

5.1.4 Percentiles

Activity 2

Problem: In how many places should a long rope be cut to make 100 equal parts?

Procedure: We know that after straightening the given rope, it should be cut into three places to make 4 equal parts. Now, mark the rope at 99 points so that each point is equidistant. Cut from the marked points.

Conclusion: By cutting at 99 points with equal distance, the rope can be divided into 100 equal parts. This is called a percentile.

Again, to further clarify the concept of percentiles, let's study the given examples. The heights (cm) of 19 students of a class are as follows.
150, 140, 132, 125, 133, 126, 123, 137, 145, 122, 149, 147, 142, 148, 136, 124, 128, 141, 144.
Write it in ascending order. Why should it be written in ascending order? Discuss.
When writing the above data in ascending order, the values are
122, 123, 124, 125, 126, 128, 132, 133, 136, 137, 140, 141, 142, 144, 145, 147, 148, 149, 150. The percentile values get larger as they increase, so it should be written in ascending order.
If the data is divided into four equal parts, what are the values that divide it into 100 equal parts? Discuss in the group. Some of these values, such as:
For example:

5th percentile (P_5): $\frac{5(19+1)}{100}^{th}$ item $= \frac{5\,(20)}{100}^{th}$ item $= 1^{th}$ item $= 122$

60th percentile (P_{60}) $= \frac{60(19+1)}{100}^{th}$ item $= \frac{60\,(20)}{100}^{th}$ item $= 12^{th}$ item $= 141$

82th percentile (P_{82}): $= \frac{82(19+1)}{100}^{th}$ item $= \frac{82\,(20)}{100}^{th}$ item $= 16.4^{th}$ item

82th percentile lies between 16th and 17th items so these are 147 and 148.

Or, $P_{82} = 147 + 0.4(148 - 147) = 147 + 0.4 = 147.4$

95th percentile (P_{95}): $P_{95} = \frac{95\,(19+1)}{100}$ item $= \frac{95\,(20)}{100}^{th}$ item $= 19^{th}$ item $= 150$

Thought Provoking Question: What are other percentile values? ....

The set of 99 values that divide any data into hundred equal parts are called percentile value. These values are denoted as $P_1, P_2, P_3, P_4, P_5 \ldots P_{99}$.

The formula for finding percentiles like quartiles is as follows.

For individual data, the position of nth percentile $(P_n) = n\left\{\frac{(N+1)}{100}\right\}^{th}$ item

Similarly, for discrete data, the position of nth percentile $(P_n) = n\left\{\frac{(N+1)}{100}\right\}^{th}$ item

Example 1

The average annual rainfall of a certain place for the past nine years is as follows:

1500 mm, 1600 mm, 1720 mm, 1850 mm, 1980 mm, 2100 mm, 2200 mm, 2220 mm, 2250 mm.

Based on this information, answer the following questions:

a. Find the 10^{th} percentile (P_{10}) and the 35^{th} percentile (P_{35}).

b. Describe the relationship between the second quartile (Q_2) and the 50^{th} percentile (P_{50}).

Solution: Here,

Writing the given data in ascending order: 1500 mm, 1600 mm, 1720 mm, 1850 mm, 1980 mm, 2100 mm, 2200 mm, 2220 mm, 2250 mm. Total number of observations: N = 9

Now by formula, a. Finding percentiles $(P_{10}) = 10\left(\frac{N+1}{100}\right)^{th}$ item $= 10\left(\frac{9+1}{100}\right)^{th}$ item

$= 1^{th}$ item = 1500 mm

35^{th} percentile $(P_{35}) = 35\left(\frac{N+1}{100}\right)^{th}$ item $= 35\left(\frac{9+1}{100}\right)^{th}$ item $= 3.5^{th}$ item

This lies between the 3^{rd} and 4^{th} items.

3^{rd} item = 1720 mm $\qquad$ 4^{th} item = 1850 mm

So, $P_{35} = \frac{1720 + 1850}{2} = 1785$ mm

b. Relationship between second quartile (Q_2) and 50th percentile (P_{50})

Second quartile (Q_2): $2\left(\frac{N+1}{4}\right)^{th}$ item $= 2\left(\frac{9+1}{4}\right) = 5^{th}$ item = 1980 mm

50^{th} percentile (P_{50}): $50\left(\frac{N+1}{100}\right)^{th}$ item $= 50\left(\frac{9+1}{100}\right)^{th}$ item $= 5^{th}$ item = 1980 mm

The values of the second quartile (Q_2) and the 50th percentile (P_{50}) are equal.

Therefore: $Q_2 = P_{50}$

Thus, the second quartile is the same as the 50^{th} percentile.

Example 2

Find the tenth percentile of the given data below:

Age (In years)	5	10	15	20	25	30
No of Students	3	7	6	2	5	7

Solution: Here,

Representing the given data into cumulative frequency table:

Age (in years)	No. of students (f)	Cumulative frequency (cf)
5	3	3
10	7	$3 + 7 = 10$
15	6	$10 + 6 = 16$
20	2	$16 + 2 = 18$
25	5	$18 + 5 = 23$
30	7	$23 + 7 = 30$
	$\sum f = N = 30$	

From the given data, sum of frequencies $(\sum f) = N = 30$

Now, according to the formula, the tenth percentile $(P_{10}) = 10\left(\frac{N+1}{100}\right)^{th}$ item

$= 10\left(\frac{30+1}{100}\right)^{th}$ item $= 3.1^{th}$ item

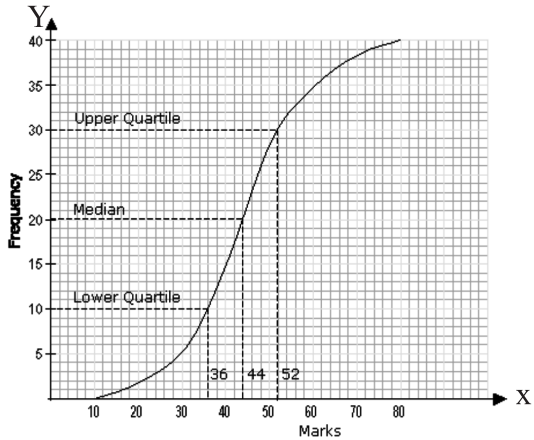
From the cumulative frequency table, the cf just greater than 3.1 is 10. So, its corresponding value is 10 so P_{10} is 10.

So, the tenth (10^{th}) percentile value 10.

Exercise 5.1

1. a. What are individual and discrete series? Clarify with examples.
   b. What is meant by quartile value? Explain with examples.
   c. What is meant by percentile value? How are they denoted? Write it.
   d. Write the formulas to find the first quartile (Q_1), second quartile (Q_2) and third quartile (Q_3) in individual data.
2. Read the above activity 1 and write the conclusion obtained from it.
3. Read the above activity 2 and write the conclusion obtained from it.
4. Write the formulas to find the first quartile (Q_1), second quartile (Q_2), and third quartile (Q_3) in discrete data.
5. Make one example each, different from the exercises and examples of indi vidual and discrete data but related to daily life.

6. Explain by observing at the graph provided.



7. What does 80% below You mean in the given picture? Explain.

8. Sabita finds the first quartile, second quartile and third quartile from the data below and says that the second quartile is the median. You also find the first quartile, second quartile and so on. Do you agree with her statement? Write with reasons.

a. 25, 48, 32, 52, 21, 64, 29, 57

b. 19, 20, 21, 23, 23, 24, 25, 27, 31

9. In a survey conducted among 543 people, it was found that the members of the following age groups watch television, on the basis of this, find the values of the first and third quartiles.

Age (In years)	20	30	40	50	60	70	80
No. of members	3	61	132	153	140	51	3

10. Last year, a survey was conducted in a village to find out how many households sold how much rice per year. The results of the survey are presented in the table below.

Crops (Pathi)	5	15	25	35	45	55
No. of household	3	7	15	5	8	2

a. Find the value of the first quartile.

b. Find the value of the third quartile.

c. Prove that the second quartile is the median.

11. The marks obtained by 19 students in Mathematics in the first quarter examination were:

30, 21, 54, 30, 37, 45, 25, 47, 37, 24,

42, 45, 33, 28, 52, 50, 47, 20, 35,

a. Find the fortieth percentile value.

b. Find the eightieth percentile value.

c. Prove that the fiftieth percentile value is equal to the second quartile.

12. The marks obtained in the annual examination for Mathematics are given on the table below, based on which answer the following questions.

Obtained marks	10	20	30	40	50	60
No. of students	3	7	15	5	8	2

a. Find the thirtieth percentile value.

b. Find the eighty-fifth percentile value.

c. Prove that the fiftieth percentile value is equal to the second quartile.

13. Construct a question with context based on discreate series. Based on the question thus constructed,

a. Find the value of the first quartile.

b. Find the value of the third quartile.

c. Prove that the second quartile is the median.

14. Construct a question with context based on individual series. Based on the question thus constructed,

a. Find the twentieth percentile value.

b. Find the sixtieth percentile value.

c. Prove that the value of the fiftieth percentile is equal to the second quartile.

Answer

1 - 7. Show to the teacher.

8. a. 26, 40, 55.75 b. 20.5, 23, 26 9. 40, 60

10. a. 25 b. 45

11. a. 33 b. 47 12. a. 30 b. 50

13 and 14 Show to the teacher. .

5.2 Dispersion of Individual and Discrete series

The prices of edible oil per liter and rice per 25 kg bag in the Nepali market for the last 11 months are as follows.

Edible oil (in rupees per liter): 180, 200, 230, 240, 245, 250, 260, 270, 300, 310, 315

Rice 25 kg (in rupees per sack): 1600, 1700, 1850, 2040, 2145, 2250, 2260, 2300, 2350, 2400, 2450

Study the above data and discuss the following questions.

a. What is the mean and median price of edible oil per liter and rice per sack?

b. How can we measure in which the given data is scattered, spread or deviated from the midpoint?

c. Write the main purpose of measuring dispersion.

In the above data, the mean price of edible oil per liter is Rs. 254.54 and the mean price of rice per 25 kg sack is Rs. 2122.27.

The median price of edible oil per liter is Rs. 250 and the median price of rice per sack is Rs. 2250. The range, interquartile deviation, mean deviation, standard deviation, and their coefficients can be calculated to measure the extent to which a given set of data is scattered or spread out or deviated from the mean. The main purpose of measuring dispersion is to find out the homogeneity or heterogeneity of a set of data.

5.2.1 Quartile Deviation and its Coeffecient

If you measure the height and weight of citizens of any country, there is certainly a huge difference. But if you measure the height and weight of players in the same country's football team, you will not find such a big difference. Yes, the quartile deviation is used to find out how much difference there is. Therefore, quartile deviation is a special piece of information that tells you how spread out the data collected is.

The ages (in years) of the people gathered in a village meeting are given below.

40, 46, 35, 50, 38, 57, 44, 52, 60, 48, 55, 56, 67, 70, 62.

Discuss in the group and find out which ages represent the quartile values (first quartile, second quartile and third quartile).

Discuss whether the given data should be written in ascending or descending order to find the quartile value, and why?

It should be written in ascending order because the values of the first quartile, second quartile and third quartile increase in the order. If it is placed in a decreasing order, then the values of the third quartile are smaller than the values of the first and second quartiles. Therefore, it is necessary to write all the given data in ascending order.

Given ages in ascending order,

35, 38, 40, 44, 46, 48, 50, 52, 55, 56, 57, 60, 62, 67, 70

The given data is individual data and has a total number of items n. = 15.

First quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{15+1}{4}\right)^{th}$ item $= \left(\frac{16}{4}\right)^{th}$ item $= 4^{th}$ item $= 44$ years

Second quartile $(Q_2) = 2\left(\frac{N+1}{4}\right)^{th}$ item $= 2 \times 4^{th}$ item $= 8^{th}$ item $= 52$ years

Third quartile $(Q_3) = 3\left(\frac{N+1}{4}\right)^{th}$ item $= 3 \times 4^{th}$ item $= 12^{th}$ item $= 60$ years

What is the difference between the third quartile (upper quartile) and the first quartile (lower quartile) called? Based on the third quartile and the first quartile, the interquartile deviation (interquartile range) is found. Thus, the difference between the third quartile (upper quartile) and the first quartile (lower quartile) is called the interquartile range. Therefore,

Semi-Inter Quartile Range (Quartile Deviation)

The half of difference between third quartile and first quartile is called quartile deviation .

Quartile deviation $= \left(\frac{Q_3 - Q_1}{2}\right)$

Inter Quartile Range

The difference between third quartile and first quartile is called inter quartile range.

Inter quartile range $= Q_3 - Q_1$

Coefficient of quartile deviation

The relative measure based on quartile deviation is called coefficient of quartile deviation.

Coefficient of quartile deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

From the example presented above,

Inter quartile range $= Q_3 - Q_1 = 60 - 44 = 16$ years

Quartile deviation $= \frac{Q_3 - Q_1}{2} = \frac{60 - 44}{2} = \frac{16}{2} = 8$ years

Coefficient of quartile deviation

$= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{60 - 44}{60 + 44} = \frac{16}{104} = 0.154$

Example 1

Find the quartile deviation and its coefficients from the given data.
16, 24, 26, 30, 32, 37, 41, 34, 45, 48, 7, 31, 39, 5, 8, 9, 11, 23, 33

Solution: Here,
When the given data is placed in ascending order,
5, 7, 8, 9, 11, 16, 23, 24, 26, 30, 31, 32, 33, 34, 37, 39, 41,45, 48
The given data is individual data and the total number of items in it is N=19.

Now, first quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{19+1}{4}\right)^{th}$ item $= \left(\frac{20}{4}\right)^{th}$ item $= 5^{th}$ item = 11

Third quartile $(Q_3) = 3\left(\frac{N+1}{4}\right)^{th}$ item $= 3 \times 5^{th}$ item $= 15^{th}$ item = 37

Quartile deviation $= \frac{Q_3 - Q_1}{2} = \frac{37-11}{2} = \frac{26}{2} = 13$

Coefficient of quartile deviation

$$= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{37-11}{37+11} = \frac{26}{48} = 0.154$$

Hence the quartile deviation and its coefficient of the given data are 13 and 0.154.

Example 2

Find the quartile deviation and its coefficient from the given data.
Distance (Km): 15, 20, 25, 40, 16, 21, 42, 35, 18, 45

Solution: Here,
The given data is in an individual data, so when the data is arranged in an ascending order,
15, 16, 18, 20, 21, 25, 35, 40, 42, 45
The total number of items n. = 10.

Now, first quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{10+1}{4}\right)^{th}$ item $= \left(\frac{11}{4}\right)^{th}$ item $= 2.75^{th}$ item

The Q_1 lies in 2^{nd} and 3^{rd} item, so

i.e. $Q_1 = 2^{nd}$ item $+ (3^{rd} - 2^{nd}) \times 0.25$

$=16 + 0.75\,(18 - 16)$

$=16 + 0.75 \times 2$

$=16 + 1.5$

$=17.5$

Third quartile: $(Q_3) = 3\left(\frac{N+1}{4}\right)^{th}$ item = 3 × 2.75th item = 8.25th item

The Q_3 lies in 8th and 9th item, so

$Q_3 = 40 + 0.25\,(42 - 40) = 40 + 0.25\,(2) = 40 + 0.5 = 40.5$

Quartile deviation = $\frac{Q_3 - Q_1}{2} = \frac{40.5 - 17.5}{2} = \frac{23}{2} = 11.5$

Coefficient of quartile deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

$$= \frac{40.5 - 17.5}{40.5 + 17.5} = \frac{23}{58} = 0.396$$

Hence, the quartile deviation and its coefficient of the given data are 11.5 and 0.396.

Example 3

The table shows the number of families and their monthly income. Find the quartile deviation and its coefficient based on this data.

Income (Rs. in thousands)	25	28	30	32	35	45
No. of family	3	4	7	6	2	1

Solution: Here,

Showing above information in cumulative frequency table

Income (Rs. in thousands)	No of family (*f*)	Cumulative frequency (*cf*)
25	3	3
28	4	7
30	7	14
32	6	20
35	2	22
45	1	23
	$\sum f = N = 23$	

Now, first quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item = $\left(\frac{23+1}{4}\right)^{th}$ item = $\left(\frac{24}{4}\right)^{th}$ item = 6th item

From above *cf* table the *cf* is just greater than 6 is 7 so its corresponding value is Rs. 28 i.e., Rs 28000 is Q_1.

Similarly, for Q_3, third quartile $= 3\left(\frac{N+1}{4}\right)^{th}$ item = 3 × 6th item = 18th item

From above cf table the cf just greater than 18 is 20 so its corresponding value is Rs. 32 i.e., Rs. 32000 is Q_3.

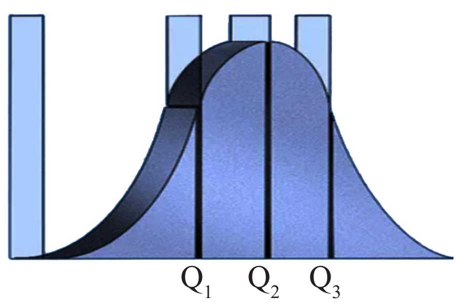
Quartile deviation = $\frac{Q_3 - Q_1}{2} = \frac{32,000 - 28,000}{2} = \frac{4,000}{2} = 2,000$

Coefficient of quartile deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{32,000 - 28,000}{32,000 + 28,000} = \frac{4,000}{60,000} = 0.067$

Hence, the quartile deviation and its coefficient of the given data are Rs 2000 and 0.067.

Exercise 5.2

1. What is dispersion? Clarify with examples.
2. What is meant by quartile deviation? Which quartiles are used to find quartile deviation? Write with reasons.
3. What is meant by interquartile range? Write.
4. If the first quartile (Q_1) = 24 and the third quartile (Q_3) = 38 in the given graph, find the quartile deviation and the coefficient of quartile deviation.



5. The lower and upper quartile values of individual data are respectively: 30 and 40.
   a. Find the quartile deviation of the data.
   b. Find the coefficient of quartile deviation.
6. The first quartile (Q_1) = 35 and the quartile deviation is 20.
   a. Find the third quartile of the data.
   b. Find the coefficient of the quartile deviation.
7. a. If the quartile deviation and its coefficient of a discrete data are 14 and $\frac{7}{20}$ respectively, find the value of its first quartile.

b. If the coefficient of quartile deviation of an individual data is $\frac{1}{4}$ and its third quartile is 15, find the first quartile value and its quartile deviation.

c. If the first quartile (Q_1) of an individual data is 12 and its quartile deviation (Q.D) is 2, find its third quartile value and its coefficient of quartile deviation.

8. The data collected at different times are given below. Based on the above data, find the quartile difference and its coefficient:

   a. Price (Rs.): 150, 200, 250, 400, 160, 210, 420

   b. Temperature (°C): 13, 40, 27, 30, 25, 22, 21, 18, 12, 13, 10.

   c. Body weight (kg): 20, 18, 25, 12, 9, 6, 21, 42, 35, 28.

   d. Student's lunch expenditure (Rs.): 50, 80, 85, 75, 70, 90, 100, 105, 120, 110, 130.

9. The weights of 28 students studying in class 9 of a school in the first week of the month of Poush are given in the table below. Based on the data, find the quartile deviation and its coefficient:

Weight (kg)	45	47	49	51	53	55
No. of students	4	8	5	3	3	5

10. The temperature data of cities located in different geographical locations but measured at the same time were found as follows. Based on the data, find the quartile deviation and its coefficient:

Temperature (ºC)	10	15	20	25	30
No. of town	3	7	10	8	2

11. The marks obtained by 39 students in the second quarterly examination of class 9 of a community school are shown in the table below. Based on the data, find the quartile deviatione and its coefficient:

Obtained marks	40	45	50	55	60	64	68
No. of students	5	7	8	6	4	6	3

12. The height of the students is measured in every quarterly examination of a school. The measured height is shown in the table below. Based on the data, find the quartile deviation and its coefficient.

Height of students (cm)	153	155	157	159	161	163	165	167	169
No. of students	8	2	4	6	3	4	7	1	4

13. The data of a survey conducted among people of different age groups who like to watch Nepali movies are given in the table below. Based on the data, find the interquartile range and its coefficient.

Age (In years)	20	30	40	50	60	70	80
No. of people	3	61	132	153	140	51	3

14. Construct a question with a context based on individual data. Based on the questions collect the data and,
    a. Find the value of the first quartile.
    b. Find the value of the third quartile.
    c. Find the quartile deviation and its coefficient.
15. Construct a question with a context based on discrete data. Based on the questions thus constructed, collect the data and,
    a. Find the value of the first quartile.
    b. Find the value of the third quartile.
    c. Find the quartile deviation and its coefficient.
    d. Why don't we need to find the value of the second quartile when finding the interquartile range and its coefficient? Give a reason.

Answer

1 - 3. Show to the teacher.

4. a. 7, b. 0.22 5. a. 5, b. 0.14 6. a. 75 b. 0.36

7. a. 26 b. 9, 3 c. 16, 0.14 8. a. 120, 0.42 b. 7, 0.35 c. 9.25, 0.45

d. 17.5, 0.189 9. 3, 0.06 10. 5, 0.25 11. 7.5, 0.14 12. 5, 0.031

13. 10, 0.2 14 - 15. Show to the teacher.

5.3 Mean Deivation and Its Coefficient

Why do we calculate the mean deviation? Doesn't the quartile deviation work? Of course not, because the quartile deviation only tells you how spread out the data is. But the mean deviation can tell you how much the values are deviated from the measures of central tendency (mean, median, and mode) are. For example, a healthy person's blood pressure should be 120/80mmHg. If it is not, health workers will ask how much it is below or above the normal range. Health workers use the mean deviation and its coefficient to get and give information about how the patient is doing now, and whether the treatment is improving or not, and how close or far it is from the mean value.

Activity-based example

Discuss the scores obtained by two friends Pemba and Tashi in mathematics exam with 25 fullmark taken 10 times during a year and discuss the questions asked.

Pemba: 17, 19, 20, 16, 22, 23, 21, 20.5, 18, 15

Tasi: 9, 10, 3, 15, 20, 16, 5, 25, 21, 23

a. What is the average score of both?

b. What are the criteria for comparing these two data?

c. When comparing the scores of two people, whose marks are far or close to the average score?

The average score obtained by Pemba:

$$= \frac{17 + 19 + 20 + 16 + 22 + 23 + 21 + 20.5 + 18 + 15}{10} = \frac{191.5}{10} = 19.15$$

Similarly, the average score obtained by Tashi:

$$= \frac{9 + 10 + 3 + 15 + 20 + 16 + 5 + 25 + 21 + 23}{10} = 14.7$$

Thus, the scores obtained by Tashi seem to fluctuate more than the average scores, while the scores obtained by Pemba seem to be relatively more stable. Thus, the mean deviation is used to compare data based on the mean or median.

> The average of the absolute values of the differences between each item from measures of central tendency such as the mean and median is called the mean deviation. The mean deviation is found based on the arithmetic mean and median. Here we calculate the mean deviation using the mean and median from the individual data and the discrete data.

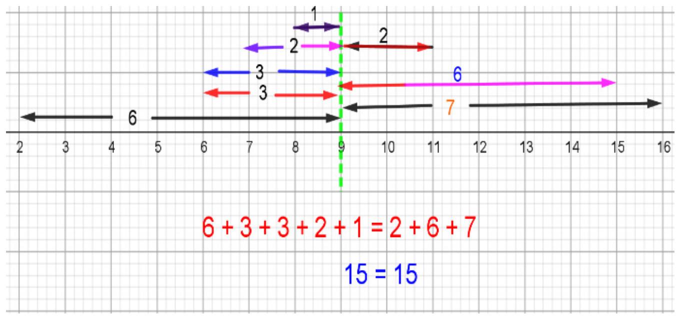
5.3.1 Mean Deviation of Individual Series

Let's take one individual data: 3, 6, 6, 7, 8, 11, 16, 15

In the first step, to find the mean deviation in individual categories, the mean must be found at first. Mean $(\overline{X}) = \frac{3+6+6+7+8+11+16+15}{8} = \frac{72}{8} = 9$

In the second step, the distance from the mean value to each given value must be found.

Value (X)	Distance from mean 9 to each data
3	6
6	3
6	3
7	2
8	1
11	2
15	6
16	7



In the third step, the mean deviation of the distance from the mean value 9 to each value should be found.

Mean Deviation (M.D.) $= \frac{6+3+3+2+1+2+6+7}{8} = \frac{30}{8} = 3.75$

So, the mean of the given data is $(\overline{X}) = 9$ and Mean Deviation (M.D.) = 3.75

Can the mean deviation be found using the formula for individual data?

Yes, of course, the mean deviation can be found using the formula, which is as follows.

If $x_1, x_2, x_3, \ldots x_n$ is an individual data.

a. Mean Deviation from Mean

i. Mean $(\overline{X}) = \frac{\sum X}{N}$

ii. Mean deviation (M.D.) $= \frac{\sum|X - \overline{X}|}{N} = \frac{\sum D}{N}$ Where $D = |X - \overline{X}|$ So the value of D is always positive and N= total number of data ,

b. Mean deviation from Median

i. Median $(M_d) = \left(\frac{N+1}{2}\right)^{th}$ item

ii. Mean deviation (M.D.) $= \frac{\sum|X - \overline{X}|}{N} = \frac{\sum D}{N}$ Where $D = |X - M_d|$. So the value of D is always positive and N = Total number of data, The mean deviation is the absolute value of the measure of dispersion. The coefficient of mean deviation is used to compare two or more categories of different unit data. The coefficient of mean deviation is a comparative measure of dispersion based on the mean deviation. The following formula is used to calculate it.

Coefficient of mean deviation (from the mean) = $\frac{\text{Mean deviation from Mean}}{\text{mean}} = \frac{\text{M.D.}}{\overline{\text{X}}}$

Coefficient of mean deviation (from the median) = $\frac{\text{Mean deviation from Mean}}{\text{mean}} = \frac{\text{M.D.}}{\text{M}_d}$

Why is it not enough to find the mean deviation? Why is the coefficient of variation also found? Discuss.

The mean deviation refers to the absolute value. The coefficient of mean deviation is found to indicate the relative value.

5.3.2 Mean Deviation of Discrete Series

How to find the mean deviation in a discrete data? Can the mean deviation be found in the same way as in individual data? Of course not, because it has frequencies, a different formula must be used. To find that, let us take a discrete series $x_1, x_2, x_3, \ldots x_n$. In which the frequencies of the respective items are $f_1, f_2, f_3, \ldots f_n$. The mean deviation can be found from the mean and the median.

a. The Mean deviation (from the mean)

(i) Mean $(\overline{X}) = \frac{\sum fX}{N}$, where $\sum fx$ = sum of product of respective item and its frequency.

(ii) Mean deviation (M.D)= $\frac{\sum f|X-\overline{X}|}{N} = \frac{\sum fD}{N}$ Where D= $|X-\overline{X}|$, So the value of D is always positive and N= total number of data,

b. Mean deviation from median

(i) Median (Md) $= \left(\frac{N+1}{2}\right)^{th}$ item

(ii) Mean Mean deviation (M.D) = $\frac{\sum f|X-M_d|}{N} = \frac{\sum fD}{N}$,
Where D = $|X - M_d|$, So the value of D is always positive and N = sum of frequency.

Study the example given above and make a mathematical exercise to clarify the concept of mean deviation and its coefficient.

Activity 1

Objectives : To build the concept on mean deviation and its coefficient
Problems : ....
Required materials : ....

Procedure : ....
Conclusion: ....

Example 1

A book has 7 pages, each page (1, 2, 3, ...) contains the following words.

The number of words on each page: 271, 296, 301, 285, 298, 327, 287

a. Find the mean deviation and its coefficient using the mean.

b. Find the mean deviation and its coefficient using the median.

Solution: Here,

Given data: X = 271, 296, 301, 285, 298, 327, 287

Total number of words (N) = 7

a. Mean deviation from mean :

According to the formula, mean $(\overline{X}) = \frac{\sum X}{N} = \frac{271 + 296 + 301 + 285 + 298 + 327 + 287}{7}$

= 295

To find the mean deviation from the mean, the following table is used.

Word (X)	$D = \lvert X - \overline{X} \rvert$
271	24
285	10
287	8
296	1
298	3
301	6
327	32
	$\sum D = 84$

Again, by using formula Mean Deviation (MD) = $\frac{\sum D}{N} = \frac{84}{7} = 12$

Coefficient of mean deviation (from the mean) = $\frac{\text{Mean deviation from Mean}}{\text{mean}}$

$= \frac{12}{295} = 0.0406$

b) Mean deviation from median,

Writing the given data in ascending order.

271, 285, 287, 296, 298, 301, 327

Now, The position of Median $(M_d) = \left(\frac{N+1}{2}\right)^{th}$ item

$= \left(\frac{7+1}{2}\right)^{th}$ item $= 4^{th}$ item

i.e. Median of the given data = 296.

Mean deviation from media can be calculated as follows:

Words (X)	$D = \lvert X - M_d \rvert$
271	25
285	11
287	9
296	0
298	2
301	5
327	31
	$\sum D = 83$

Again, by using formula Mean deviation (MD)= $\frac{\sum D}{N} = \frac{83}{7} = 11.85$

Coefficient of mean deviation (from the median) = $\frac{\text{Mean deviation from Median}}{\text{Median}} = \frac{11.85}{296} = 0.04$

Example 2

The marks obtained by a student in the first trimester examination of Mathematics of 75 whole marks in a school are given below.

Obtained marks	10	15	20	25	30	35	40
No. of Students	8	10	5	3	5	2	7

a. Find the mean deviation and its coefficient using the mean.

b. Find the mean deviation and its coefficient using the median.

Solution: Here,

a. To find the mean deviation from the mean, the following table is used, placing the data in ascending order.

Obtained marks (x)	No. of students (f)	fx	$D = \lvert X - \overline{X} \rvert$	fD
10	8	80	12.625	101
15	10	150	7.625	76.25
20	5	100	2.625	13.125
25	3	75	2.375	7.125
30	5	150	7.375	36.875
35	2	70	12.375	24.75
40	7	280	17.375	121.625
	$\sum f = N = 40$	$\sum fx = 905$		$\sum fD = 380.75$

Now, by using formula, Mean $(\overline{X}) = \frac{\sum fX}{N} = \frac{905}{40} = 22.625$

Again, Mean deviation (M.D) $= \frac{\sum D}{N} = \frac{380.75}{40} = 9.52$

Coefficient of MD $= \frac{\text{MD from Mean}}{\overline{X}} = \frac{9.52}{22.625} = 0.4207$

b. To find the mean deviation from the median, the following table is used, placing the data in ascending order.

Obtained marks (x)	No. of students (f)	cf	$D = \lvert X - M_d \rvert$	fD
10	8	8	10	80
15	10	18	5	50
20	5	23	0	0
25	3	26	5	15
30	5	31	010	50
35	2	33	15	30
40	7	40	20	140
	$\sum f = N = 40$			$\sum fD = 365$

Now, By using formula, Median $(M_d) = \left(\frac{N+1}{2}\right)^{th}$ item

$= \left(\frac{40+1}{2}\right)^{th}$ item $= \left(\frac{41}{2}\right)^{th}$ item $= (20.5)^{th}$ item

Now, By using formula, Median $(M_d) = \frac{\sum fD}{N} = \frac{365}{40} = 9.125$

Coefficient of MD $= \frac{\text{MD from Median}}{\text{Median}} = \frac{9.125}{20} = 0.456$

Exercise 5.3

1. What is meant by mean deviation? Write it.
2. How can mean deviation be found? Write it.
3. Where is mean deviation used in our daily life? Find it.
4. Why is mean deviation considered more reliable than quartile deviation? Write with reasons.
5. Why is it not enough to find mean deviation, why is the coefficient of variation also found?
6. Explain how mean deviation is found from the mean.
7. Explain how the mean deviation is found from the median.

8. Various data collected at different times are given below. Based on the data, find the mean or average deviation from the mean and its coefficient.
   a. Price (Rs.): 150, 200, 250, 400, 160, 210, 422
   b. Temperature (°C) 20, 18, 25, 12, 9, 6, 22, 45, 35, 28.
   c. Body weight (kg): 13, 40, 27, 30, 25, 22, 21, 18, 12, 13, 10.
   d. Student's lunch expenses (Rs.): 50, 80, 85, 75, 70, 90, 100, 105, 120, 110, 127.
   e. Rainfall (mm): 14, 10, 8, 12, 22, 28, 16, 24, 26.
   f. Scores: 17, 10, 15, 7, 13, 9, 6, 18, 11, 14.
9. Based on the given data, find the mean deviation from the median and its coefficient.
   a. Price (Rs.): 150, 200, 250, 400, 160, 210, 422
   b. Temperature (°C): 20, 18, 25, 12, 9, 6, 22, 45, 35, 28
   c. Body weight (kg): 13, 40, 27, 30, 25, 22, 21, 18, 12, 13, 10
   d. Student's lunch expenses (Rs.): 50, 80, 85, 75, 70, 90, 100, 105, 120, 110, 127
   e. Rainfall (mm): 14, 10, 8, 12, 22, 28, 16, 24, 26
   f. Scores: 17, 10, 15, 7, 13, 9, 6, 18, 11, 14
10. The first 10 natural numbers are given: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
   a. Find the mean deviation from the mean and its coefficient.
   b. Find the mean deviation from the median and its coefficient.
   c. Is there any difference in the value of the mean deviation from the mean and the median? Write with reasons.
11. The marks obtained by the students of class 9 of a school in the second terminal examination in Science and Technology are shown in the table below.

Obtained marks	40	45	50	55	60	64	68
No. of students	5	7	8	6	4	6	3

   a. In which series the given data is presented?
   b. Find the total number of students' marks given in the table.
   c. Find the mean deviation and its coefficient from the mean of the data.
   d. Find the mean deviation and its coefficient from the median of the data.
12. The temperature recorded in a day at a certain place is shown in the table below.

Temperature (°C)	20	25	28	29	33	38	42	43
Frequency	6	20	24	28	15	4	2	1

   a. In which series is the given data presented?
   b. Find the total frequency (Σf).
   c. Find the mean deviation from the mean of the given data and its coefficient.

d. Find the mean deviation from the median of the given data and its coefficient.

13. The rainfall recorded in mm at a certain place during five months (Jestha, Ashadh, Shrawan, Bhadra, Ashoj) is shown in the table below.

Rain (mm)	20	25	30	35	40
Day	5	8	12	10	5

a. Write the formula to find the median of discrete data.

b. Find the mean from the given data.

c. Find the median from the given data.

d. Find the mean deviation from the mean of the given data and its coefficient.

e. Find the mean deviation from the median of the given data and its coe ficient.

Answer

1 - 7. Show to the teacher. 8. a. 88.57, 0.34 b. 9, 0.44 c. 7.09, 0.337 d. 18.54, 0.20 e. 6.42, 0.36 f. 3.4, 0.28 9. a. 80.28, 0.38 b. 9, 0.42 c. 7.09, 0.337 d. 18.36, 0.204 e. 6.22, 0.34 f. 4, 0.33 10. a. 2.5, 0.45 b. 2.5, 0.45 c. no need

11. a. Discrete series b. 39 c. 7.59, 0.14 d. 7.51, 0.15 12. a. Discrete series b. 100 c. 2.94, 0.102 d. 2.94, 0.101 13. a. $\overline{X} = \frac{\sum fX}{N}$ b. 30.25 c. 30 d. 4.81, 0.159 e. 4.75, 0.158

5.4 Standard Deviation and Its Coefficient

The concept of standard deviation was discovered by Karl Pearson in 1893. Why is it used more than other deviations? Is it more reliable and trustworthy than other deviations? Discuss.

In a series or distribution of data, the square root of the average of each item and the square of the mean deviation is called the standard deviation. In other words, the standard deviation is calculated by squaring the deviation of each item of any data and dividing their sum by the total number of items and taking the square root. Therefore, it is also called the square root of the average of the square of the mean deviation. It is denoted by the Greek letter σ (sigma.. Standard deviation measures the absolute spread or dispersion of any data. It only determines the uniformity of the distribution of the data. The smaller the value of standard deviation, the greater the degree of uniformity in the data. Therefore, its calculation gives information about how the distribution mean of the data represents that data.

5.4.1 Standard deviation on Individual Data

If $x_1, x_2, x_3, ..., x_n$ are values of an individual data, then its standard deviation can be calculated using different methods.

Now, let us discuss the different methods.

a. Actual Mean Method) : In this method, to find the standard deviation, the actual mean $(\overline{X})$ must be calculated.

If the mean is a decimal value, the calculation may become somewhat difficult.

The following formula is used to calculate the standard deviation using this method:

$$(S.D) = \sqrt{\frac{\Sigma(X-\overline{X})^2}{N}} = \sqrt{\frac{\Sigma d^2}{N}}$$

$d = X - \overline{X}$: the difference between each item of the data and the mean

N = total number of items,

$\overline{X}$ = Mean value of the data

> Steps for Calculating Standard Deviation by the Actual Mean Method
>
> 1. Find the mean $(\overline{X})$ using the formula: Mean $(\overline{X}) = \frac{\Sigma X}{N}$
> 2. Find the deviation of each given value X from the mean $\overline{X}$: $d = X - \overline{X}$.
> 3. Then, find the square of each deviation d^2 and calculate the total $\sum d^2$.
> 4. Finally, calculate the standard deviation using the formula: $(S.D) = \sqrt{\frac{\sum d^2}{N}}$.

b. Direct Method : In this method, actual mean and assumed mean are not required to calculate standard deviation.

The sum of all given values X is calculated, and each value is squared to find ΣX^2.

Then the standard deviation is calculated using the following formula:

$$(S.D.) = \sqrt{\frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2}$$

X = the value of any item N = total number of items

c. Assumed Mean Method

It is difficult and time-consuming to calculate the standard deviation using the actual mean when the values are large or complicated.

In such a situation, one value from the data is taken as the assumed mean, and the standard deviation is calculated based on it.

To find the standard deviation by this method, the given data is first written in ascending order, and a central value is taken as the assumed mean.

> **Steps for Calculating Standard Deviation by the Assumed Mean Method**
>
> 1. Choose a central value from the given data as the assumed mean A.
> 2. Find the deviation of each value from the assumed mean: $d = X - A$.
> 3. Find the sum of deviations Σd.
> 4. Calculate the square of deviations d^2 and find Σd^2.
> 5. Apply the formula of standard deviation:
> $(S.D) = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d^2}{N}\right)}$

In the assumed mean is denoted by A.

The formula used is: Standard Deviation (S.D.) = $\sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2}$

Where, d = X – A = the deviation of each item from the assumed mean or actual mean A = assumed mean   N = total number of items   X = value of any item

5.4.2 Standard Deviation in a Discrete (Ungrouped) Frequency Distribution

How do we calculate the standard deviation in a discrete frequency distribution? Can it be calculated in the same way as in individual data?

Certainly not-because in discrete data, values repeat due to frequency, so a different formula must be used.

To understand this, consider discrete data with values

$x_1, x_2, x_3, \ldots x_n$ and corresponding frequencie $f_1, f_2, f_3, \ldots f_n$

Standard deviation can be calculated using different methods:

a. Actual Mean Method: To calculate standard deviation using this method, the following formula is used:

$$(\text{S.D. or } \sigma) = \sqrt{\frac{\sum f(X-\bar{X})^2}{N}} = \sqrt{\frac{\sum f d^2}{N}}$$

Where, $d = X - \bar{X}$ = the deviation of each item from the mean, N = Σf = total frequency, X = any item in the data.

Note: Using this method becomes troublesome if the actual mean is a decimal, because calculations become lengthy.

Hence, the direct method is usually more suitable.

b. Direct Method: To calculate standard deviation by the direct method, use:

$$(\text{S.D. or } \sigma) = \sqrt{\frac{\sum f X^2}{N} - \left(\frac{\sum fX}{N}\right)^2}$$

Where, X = any value and Σf = N = total frequency

c. Assume Mean Method: In this method, the actual mean is not calculated. Instead, a central value of the given data is taken as the assumed mean.

Use the following formula:

$$(\text{S.D. or } \sigma) = \sqrt{\frac{\sum f d^2}{N} - \left(\frac{\sum f d}{N}\right)^2}$$

Where, $d = X - A$ = deviation of each item from the assumed mean

A = assumed (imaginary) mean,

Σf = N = total frequency, X = any value

Steps for Calculating Standard Deviation Using the Assumed Mean Method

1. Choose a central value from the data as the assumed mean A.
2. Find the deviation between each value hand the as sumed mean A: $d = X - A$
3. Multiply the frequency f with each deviation d to obtain fd.
4. Calculate the product of frequency and squared deviation fd^2.
5. Finally, apply the formula: (S.D) = $\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd^2}{N}\right)}$

5.4.3 Coefficient of Standard Deviation

You already know what standard deviation is. But what do we understand by the coefficient of standard deviation?

How is it calculated?

What is the difference between standard deviation and its coefficient?

Let's discuss and reach a conclusion. There is a difference between them: Standard deviation is an absolute measure of dispersion. The coefficient of standard deviation is a relative measure based on standard deviation.

To find coefficient of standared deviation we use following formula.

Coefficient of Standard Deviation = $\frac{\sigma}{X}$

Where: σ = standard deviation, X = mean (average)

A smaller coefficient of standard deviation means greater uniformity or stability in the data distribution. Conversely, a larger coefficient means less uniformity or more dispersion.

Therefore, while comparing different data sets, the data with the smaller coefficient of standard deviation is considered better (more uniform).

5.4.4 Coefficient of Variation

Standard deviation is an absolute measure of dispersion.

The relative measure of dispersion associated with standard deviation is called the coefficient of variation (CV). Since the coefficient of standard deviation is often a very small decimal, it is usually multiplied by 100 and expressed as a percentage. This percentage value is called the coefficient of variation.

The coefficient of variation is a comparative measure and is used to compare the uniformity or dispersion among two or more data sets.

If the coefficient of variation is large, the data distribution has greater dispersion.

If the coefficient of variation is small, the data distribution has greater uniformity or stability. The coefficient of variation is denoted by C.V..

The formula is: Coefficient of Variation (C.V) = $\frac{\sigma}{X} \times 100\%$ Where: σ = standard deviation and $\overline{X}$ = mean (average)

Example 1

While going to a school, a group of ten students walking on the road were asked, “How much money do you have in your pockets?”

It was found that they had the following amounts of money (in rupees):

5, 10, 15, 20, 25, 30, 35, 40, 45, 50

Based on this data, find the standard deviation, the coefficient of standard deviation, and the coefficient of variation.

Solution: Here, given data X = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50

Total number of items (N) = 10

X	$d = X - A$	d^2
5	–25	625
10	–20	400
15	–15	225
20	–10	100
25	–5	25
30	0	0
35	5	25
40	10	100
45	15	225
50	20	400
	$\sum d = -25$	$\sum d^2 = 2125$

Assumed mean A : 30 (for example)

Note: Any convenient value of X may be taken as A.

According to the formula, the standard deviation (using assumed mean) is

$$(\text{S.D}) = \sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2}$$

$$(\text{S.D}) = \sqrt{\frac{2125}{10} - \left(\frac{-25}{10}\right)^2} = \sqrt{212.5 - 6.25} = \sqrt{206.25} = 14.36$$

$$(\overline{X}) = A + \frac{\sum d}{N} = 30 + \left(\frac{-25}{10}\right) = 30 - 2.5 = 27.5$$

$$\textbf{Coefficiant of S.D.} = \frac{\sigma}{\overline{X}} = \frac{14.36}{27.5} = 0.5221$$

$$\textbf{Coeficient of variation (C.V)} = \frac{\sigma}{\overline{X}} \times 100\% = \frac{14.36}{27.5} \times 100\% = 52.21\%$$

Discussion question: Can this problem be solved by another method?

Example 2

In Class 9 of Sanskrit Secondary School, a compulsory mathematics class test (out of 25 marks) was conducted for 18 students. The marks they obtained in the test are given in the table below.

Obtained marks	6	10	12	14	24
Number of students	2	3	4	5	4

Based on the given data,

a. Write the formula to calculate standard deviation using the assumed (or imaginary) mean method.
b. Calculate the standard deviation.
c. Find the coefficient of standard deviation.
d. Calculate the standard deviation and the variance and explain the difference between them.
e. Find the coefficient of variance.

Solution: Here, assumed (imaginary) mean A = 12

Obtained Marks (x)	Number of students (f)	$x - A = d$	fd	fd^2
6	2	6 – 12 = –6	– 12	72
10	3	10 – 12 = –2	– 6	12
12	4	12 – 12 = 0	0	0
14	5	14 – 12 = 2	10	20
24	4	24 – 12 = 12	48	576
	$\sum f = N = 18$		$\sum fd = 76$	$\sum fd^2 = 680$

Assumed mean A= 12

a. The formula to calculate standard deviation using the assumed (imaginary) mean method is as follows:

$$(\sigma) = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

b. According to the formula, $(\sigma) = \sqrt{\frac{680}{18} - \left(\frac{76}{18}\right)^2} = \sqrt{37.778 - 17.827} = \sqrt{19.95} = 4.47$

c. According to the formula, the actual mean is: $(\overline{X}) = A + \frac{\sum fd}{N} = 12 + \frac{76}{18} = 16.22$

The coefficient of standard deviation is: Coeff.of SD = $\frac{\sigma}{\overline{X}} = \frac{4.47}{16.22} = 0.275$

The standard deviation is: $(\sigma) = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$

d. From the above: Standard deviation $(\sigma) = 4.47$

variation $= \sigma^2 = (4.47)^2 = 19.9809$

The square of the standard deviation gives variance.

e. The coefficient of variation (relative measure of dispersion) is: $(C.V) = \frac{\sigma}{\overline{X}} \times 100\%$

$$= \frac{4.47}{16.22} \times 100\% = 27.5\%$$

Exercise 5.4

1. a. What is meant by standard deviation.
   b. Write the different methods by which standard deviation can be calculated.
   c. Write all the formulas used to find standard deviation in individual and discrete data using different methods.
   d. Explain clearly how standard deviation is calculated.
   e. Find out and write where standard deviation is used in our daily life.
   f. What is meant by the coefficient of standard deviation? Write.
2. a. Write the definition of coefficient of variation and explain how it is calculated.
   b. Explain how having a higher or lower coefficient of variation can help in analyzing data.
3. The following data have been collected at different times for different purposes. Based on the data, calculate the standard deviation, its coefficient, and the coefficient of variation.
   a. Price (Rs.): 150, 200, 250, 400, 160, 210, 420
   b. Temperature (°C): 20, 18, 25, 12, 9, 6, 21, 42, 35, 28
   c. Body weight (kg): 13, 40, 27, 30, 25, 22, 21, 18, 12, 13, 10
   d. Students' snack expense (Rs): 50, 80, 85, 75, 70, 90, 100, 105, 120, 110, 130
   e. Rainfall (mm): 14, 10, 8, 12, 22, 28, 16, 24, 26
   f. Marks obtained: 17, 10, 15, 7, 13, 9, 6, 18, 11, 14
4. The daily wages of employees working in a supermarket are as follows:

Wage (Rs.)	1200	1300	1400	1500	1600
Number of employees	8	12	15	9	6

   a. Find the average wage of the employees.
   b. Find the standard deviation of this data.
   c. Find the coefficient of variation of this data.
5. Based on the following data, find the standard deviation, its coefficient, and the coefficient of variation:

a.

Age (In years)	10	12	13	14	15	16
Number of children	8	12	15	9	6	5

b.

Obtained marks	35	45	50	55	60	65	70	75
Number of students	8	4	5	5	6	6	6	1

c.

Daily Wages (Rs.)	1000	1050	1100	1150	1200
Number of employees	14	9	5	8	5

Answer

1 - 2. Show to the teacher.

3. a. 102.38, 0.401, 40.04% b. 10.76, 0.498, 49.83% c. 8.71, 0.415, 41.48%
   d. 22.40, 0.243, 24.27% e. 6.96, 0.39, 39.14% f. 3.87, 0.32, 32.27%
4. a. 1386 b. 123.30 c. 8.89%
5. a. 1.71, 0.132, 13.20% b. 12.30, 0.226, 22.6% c. 71.65, 0.066, 6.65%

5.3 Box and Whisker

The inventor of the box plot is the mathematician, John Tukey. It is also called a Box and Whisker Plot. Specifically, the whiskers in a box plot represent the smallest and largest values in the data at the two ends of the box.

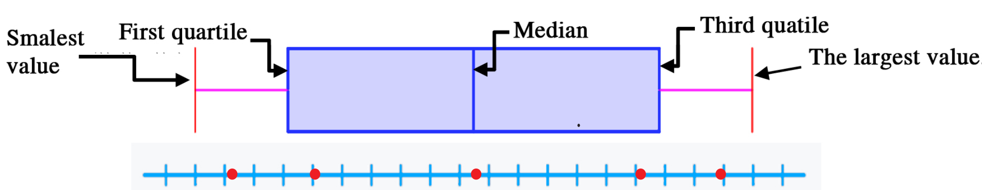
Here, we will study: What is box plot? How to make a box plot from given data? How to analyze data using a box plot ? and how to compare and interpret multiple box plots?

5.4 Whisker Box Plot

A box plot is a graphical representation from which we can identify: The smallest value in the data, the largest value, the lower quartile (Q_1), the median (M_d), and the upper quartile (Q_3).

Thus, the five-number summary of a data set consists of: smallest value, first quartile (Q_1), median (M_d), third quartile (Q_3), and largest value.

To clearly show these values, the box plot is drawn on a scale, with each of the five numbers marked appropriately on the diagram.

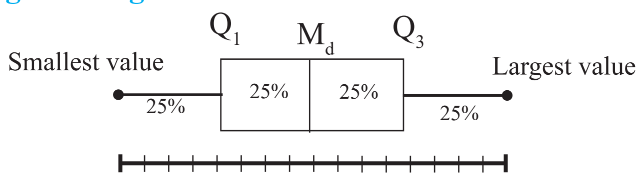


A box plot is very useful for comparing and analyzing two or more data sets. It is especially convenient for graphically comparing the median, spread, and quartiles.

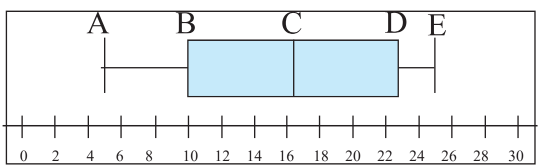
The five key values (smallest value, Q_1, median, Q_3, largest value) are required to draw a box plot.

These values divide the data as shown in the diagram below.

5.3.1 Study the given diagram



a. In the diagram, a box and whisker plot is shown.



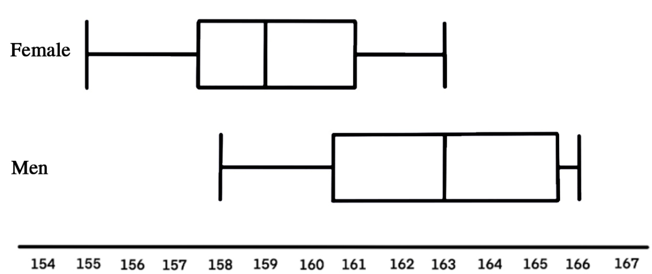
i. What do the English letters A, B, C, D, and E in the diagram represent?

ii. What are the smallest and largest values in the data?

iii. What is the lower quartile (Q_1), median (M_d), and upper quartile (Q_3)?

a.	A = ·····	B = ·····	C = ·····	D = ·····	E = ·····
b.	smallest values = ···	largest values = ·····			
c.	Q_1 = ·····	median = ·····	Q_3 = ·····		

b. In the diagram, two box and whisker plots are shown. The first box plot represents the height of women (in Centimeters). The second box plot represents the height of men (in Centimeters). Study these box plots carefully and discuss the questions asked below.



i. Compare the heights of women and men.

(ii) What is the maximum height of women? Write it.

(iii) If a person's height is 162 cm, is it more likely that the person is a woman or a man? Explain your reasoning.

By studying the above whisker box plots, we can see that: The upper value and median height of men are higher than those of women. This indicates that men are taller than women on average. The maximum height of women is 163 cm, which means the tallest woman is 163 cm. If a person's height is 162, we can determine whether they are likely a man or woman by comparing it to both box plots. In the second box plot (men), there are many men with height 162 cm or more, so a person with height 162 cm is most likely a man.

Steps to construct a box plot

1. Find the median and quartiles of the data.
2. Draw a scale on the graph and mark the five key values: smallest value, lower quartile (Q_1), median (M_d), upper quartile (Q_3), largest value
3. Draw the box by connecting Q_1 and Q_3 and draw horizontal lines at the mini mum and maximum values (whiskers).

Example 1

Using the given data, construct a whisker box plot.

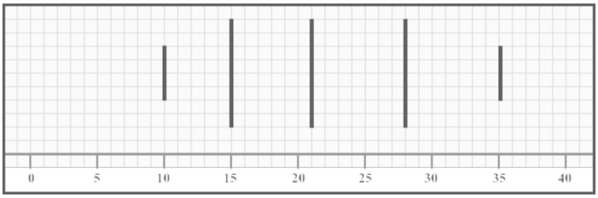
Smallest value	10
Lower quartile (Q_1)	15
Median (M_d)	21
Upper quartile (Q_3)	28
Largest value	35

Solution

Here, the five figure summary are as follows: smallest value: 10, lower quartile (Q_1): 15, median (M_d): 21, upper quartile (Q_3): 10, largest value: 35

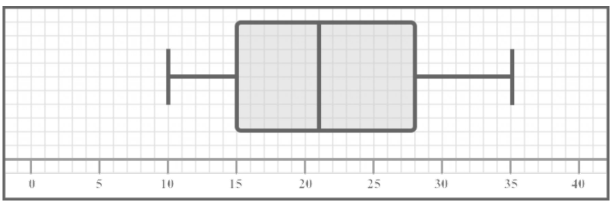
Draw a scale and mark these values on the graph.

While drawing the scale, ensure it accommodates the smallest and largest values. Draw vertical lines at these five points as shown in the graph below.



On the graph, the horizontal line is taken from 0 to 40.

In the box, connect the lower quartile ($Q_1 = 15$) to the upper quartile ($Q_3 = 28$). The lower whisker extends to the smallest value = 10, and the upper whisker extends to the largest value = 35.



The box is drawn by connecting Q_1 and Q_3, and horizontal lines are drawn at the smallest and largest values.

Example 2

Draw a box and whisker plot from the following data and present it on a graph:

1, 2, 1, 3, 5, 7, 15, 8, 10, 12, 7

Solution: Here, 1, 1, 2, 3, 5, 7, 7, 8, 10, 12, 15

Writting the data in ascending order:1, 1, 2, 3, 5, 7, 7, 8, 10, 12, 15

Total number of items n. = 11

Five figure summary: Smallest = 1 and Largest = 15

Lower quartile (Q_1) position: $\left(\frac{N+1}{4}\right)^{th}$ item = $\left(\frac{11+1}{4}\right)^{th}$ item= 3^{rd} item = 2

Therefore, $Q_1 = 2$

Median (M_d) position: $\left(\frac{N+1}{2}\right)^{th}$ item = $\left(\frac{11+1}{2}\right)^{th}$ item= 6^{th} item = 7

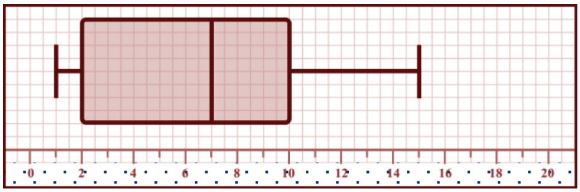
Therefore, $M_d = 7$

Upper quartile (Q_3) position: $3\left(\frac{N+1}{4}\right)^{th}$ item = $3\left(\frac{11+1}{4}\right)^{th}$ item = $3\left(\frac{12}{4}\right)^{th}$ item

$= 3 \times 3^{th}$ item = 9^{th} item = 10

Therefore, upper quartile (Q_3) = 10

The whisker box plot using the five-figure summary is shown below:

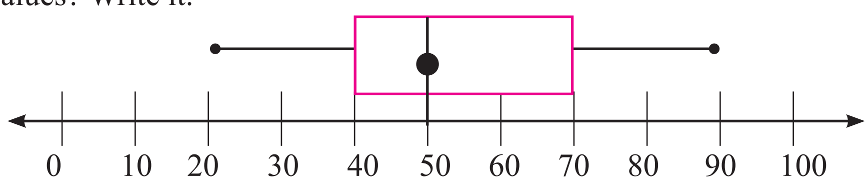


Smallest value	1
Lower quartile (Q_1)	2
Median (M_d)	7
Upper quartile (Q_3)	10
Largest value	15

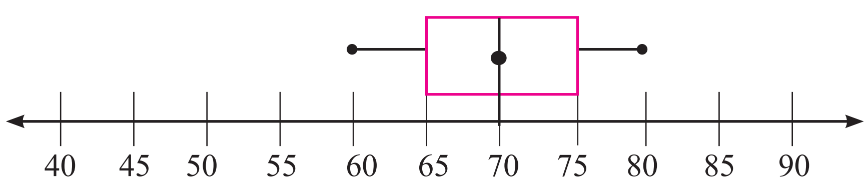
The five key values are marked with vertical lines. The box is drawn by connecting the lower quartile (Q_1 = 2) to the upper quartile (Q_3 = 10). The lower whisker extends to the Smallest value = 1, and the upper whisker extends to the Largest value = 15.

Exercise 5.5

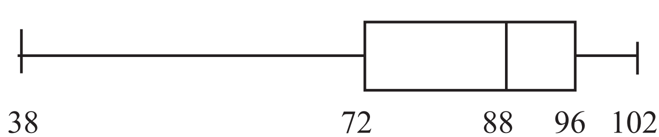
1. What are the five-figure summary values, while constructing box plot. From the given box plot find the five figure summary and what are their respective values? Write it.



2. In the given box and whisker plot, the heights (in inches) of students in a secondary school basketball team are shown. What are the five-number summary values, and what are their respective values? Write them.



3. In the given box and whisker plot, the weights (in kg) of students studying in a secondary school are shown. Based on this box and whisker plot, answer the questions given below.



a. What is the maximum weight?

b. What is the minimum weight?

c. What is the median weight?

d. What percentage of students weighs more than 72 kg?

e. What percentage of students weighs between 88 kg and 96 kg? Write your answer.

f. Do you expect the mean to be higher or lower than the median? Explain your reasoning.

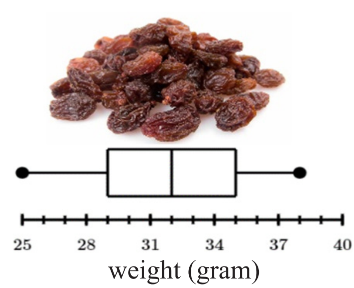
4. In Baishakh 2080, the weights of 14 women were recorded as follows. Draw a box and whisker plot and present it on a graph:

190, 175, 187, 199, 205, 187, 176, 180, 187, 191, 200, 193, 188, 196

5. The scores of 15 students in Class 9 in the first term exam of mathematics of 75 mark are as follows. Draw a box and whisker plot and present it on a graph:

61, 55, 54, 69, 74, 65, 58, 73, 71, 60, 66, 70, 56, 64, 73

6. The graph shows the weight of a dried dates. Now answer: What percentage of dates weighs more than 29 grams? Can this be determined by looking at the whisker box plot?



Answer

1. a. The smallest value = 20, lower quartile (Q_1) = 40, median = 50, upper quartile (Q_3) = 70, largest value = 90

2. a. The smallest value = 60, lower quartile (Q_1) = 65, median = 70, upper quartile (Q_3) = 75, largest value = 80

3. a. 102 b. 38 c. 88 d. 75% e. 25%

4 - 5. Draw whisker box plot and show to the teacher. 6. 75%, yes

Project Work

1. **Problem:** Collect the marks obtained by your classmates in Mathematics and Social Studies in a unit test. Based on the collected data, calculate: Class average marks, Quartile deviation (QD), Mean deviation (MD), Standard deviation (SD). Compare these measures of dispersion and determine which one is most appropriate to use. Present your statements with conclusion.

Procedure: Ask all classmates for their marks in Mathematics and Social Studies and record them. Organize the collected marks subject-wise in a table. Using the necessary formulas, calculate the following:

1. Which is the average value to be calculated? Justify your choice.
2. Find Quartile Deviation (QD) and its coefficient
3. Find Mean Deviation (MD) and its coefficient
4. Find Standard Deviation (SD) and its coefficient
5. Interpret the results

Uses and Importance: Which subject's data shows more dispersion? How does higher dispersion affect learning outcomes? How can quartile deviation, mean deviation, and standard deviation be applied in different areas? Analyze and explain their importance.

Conclusion: Among QD, MD, and SD, which measure is more reliable and trustworthy? Which subject had more effective learning? Draw a conclusion.

Reflection: How reliable are your study method and conclusion? Did you reach a correct conclusion? What difficulties did you face? Which method was most appropriate for your conclusion? Could you have reached the same conclusion using another method? Could graphs or other techniques have helped? How can this be applied in daily life? Reflect on these points.

2. **Problem:** Collect the weights of students in a class, separately for boys and girls. Based on the data: Calculate the average weight for boys and girls separately. Compare the average weights. Calculate and compare quartile deviation, mean deviation, and standard deviation. Determine which measure of central tendency or dispersion is most appropriate. Finally, prepare your research report and present it in the classroom using: Chart paper, PowerPoint, or any other suitable method.

Lesson 6

Limit

Zeno
(490 – 425B.C.)

The famous Italian philosopher Zeno presented the fact that there is no motion when space and time are divided into infinite pieces on a straight line, which is also known as Zeno's Paradox. In the sixteenth century around, mathematicians such as Newton, Leibniz, and later Cauchy developed the concept of limit values.

The concept of limit values was developed from the major problems of how to find the area of geometric figures with curves and how to draw a tangent line at a point on any curved line. For the study of limit values, prior knowledge of subjects such as algebra, coordinate geometry and trigonometry is required.

6.1 Concept of Infinity

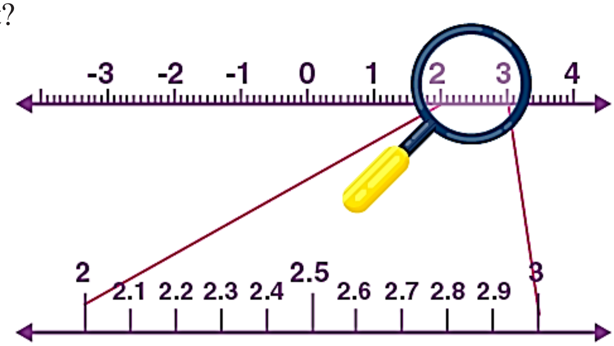
Thought Provoking Question:

a. Write the set of natural numbers using the listing method. Are natural numbers a finite set?

b. How many points are there on a straight line?

c. Divide 1 by 3. How many times does the division operation end? Does the quotient end in a terminating decimal?

d. What will be the final values when the number line below is extended to the right and left?



Infinity comes from the Latin word 'Infinitas'. The word Infinitas is made up of two words in and finitas, in which in means no or not, while the root word finis of finitas means end and bounds. Thus, literally, an object and process that does not end and has no limits is called infinity. This is symbolized by ∞. Infinity is not a number in itself, it is a concept of infinity or limitlessness. On the number line, real number approaches '+ ∞' to the right and '-∞' to the left. But "+ ∞" and "–∞" are not exactly.

6.1.1 Division by Zero

Activity 1

Calculate the values given below.

For example: $\frac{5}{2} = 2.5$,

$\frac{5}{1} = ...,\ \frac{5}{0.1} = ...,\ \frac{5}{0.01} = ...,\ \frac{5}{0.001} = ...,\ \frac{5}{0.0001} = ...,\ \frac{5}{0.00001} = ...,\ \frac{5}{0.000000001} = ...$

Similarly, if the denominator is a number that is very close to 0, what will be its value? Estimate. What can you write for the value of $\frac{5}{0}$? Guess.

When any number is divided by 0, the quotient is undefined because it cannot be defined in a specific way. This can also be said to be the value of the function being infinite. We need this concept especially to study the continuity of a function.

$\therefore \frac{a}{0} = \infty$ (Undefined), where $a \neq 0$,

Example 1

In a function $f(x) = \frac{1}{x-2}$, find the value of $f(x)$ by putting the value of $x = 1, 2$ and 3. At what value of x is the function undefined? Find it.

Solution

Given function, $f(x) = \frac{1}{x-2}$

When $x = 1$ then, $f(1) = \frac{1}{1-2} = \frac{1}{-1} = -1$

When $x = 2$ then, $f(2) = \frac{1}{2-2} = \frac{1}{0} = \infty$ (undefined)

When $x = 2$ then, $f(3) = \frac{1}{3-2} = \frac{1}{1} = 1$

$\therefore x = 2$ then the given function is undefined.

Exercise 6.1

1. What is infinity? Do integers extend to infinity?
2. Fill in the blanks below.

   a. $\infty + \infty = ...$ $\quad (-\infty) + (-\infty) = ...$

   b. $(+\infty) \times (+\infty) = ...$ $\quad (-\infty) \times (-\infty) = ...$ $\quad (-\infty) \times (\infty) = ...$

c. If x is a whole number

i. $x + (-\infty) = \ldots$ ii. $x - \infty = \ldots$ iii. $x + \infty = \ldots$ iv. $x - (-\infty) = \ldots$

v. $x \times \infty = \ldots$, for $x > 0$ vi. $x \times (-\infty) = \ldots$, for $x > 0$

vii. $x \times \infty = \ldots$, for $x < 0$ viii. $x \times (-\infty) = \ldots$, for $x < 0$

3. The function $f(x) = \dfrac{5}{x^2 - 1}$ is given.

a. Find the value of $f(x)$ when $x = 0, -1, 1$ and 2.

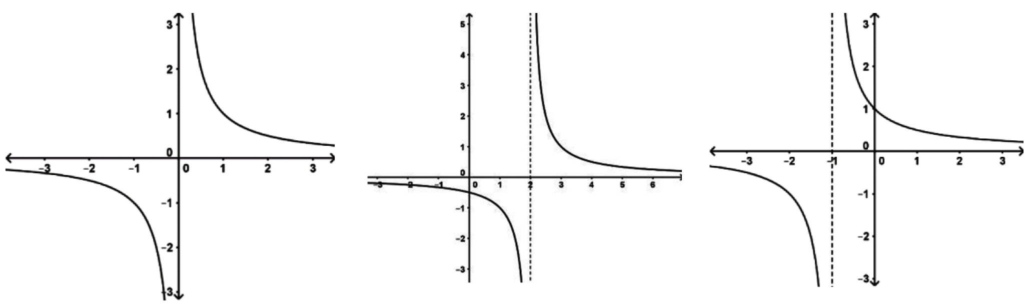
b. For what value of x the function is undefined? Write it.

4. The function $f(x) = \dfrac{1}{x^2 - 5x + 6}$ is given,

a. Find the value of $f(x)$ when $x = 0, 1, 2$ and 3.

b. For what value of x the function is undefined? Write it.

5. The graph of the function $f(x)$ is presented below. Based on the graph, write at which point the function is undefined.



Answer

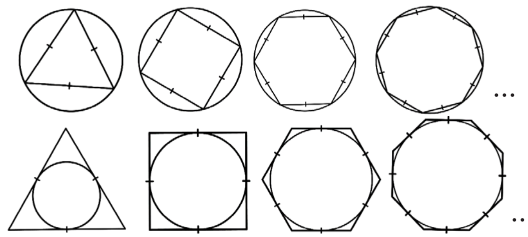
1. Show to the teacher. 2. a. $\infty, -\infty$ b. $\infty, \infty, -\infty$ c. i. $-\infty$, (ii) $-\infty$, (iii) ∞, (iv) ∞, (v) ∞, (vi) $-\infty$, (vii) $-\infty$, (viii) ∞ 3. a. -5, undefined, $\frac{5}{3}$ b. $x = -1, 1$

4. a. $\frac{1}{6}$, $\frac{1}{2}$, undefined b. $x = 2, 3$ 5. $x = 0, x = 2, x = -1$

6.2 Concept of Limit

Activity 1

Study the figures below and discuss the questions:



a. What is a regular polygon?

b. In the figure, if the number of sides of the regular polygon inside the circle is increased to infinity, what is the difference in area between the polygon and the circle? Estimate. Which shape does the regular polygon come closest to?

c. What is the limiting condition of a regular polygon?

d. What is the limiting condition of the perimeter of a regular polygon?

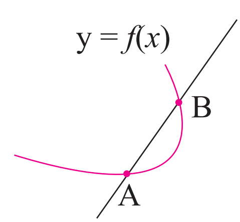
e. What is the limiting condition of the area of a regular polygon?

When a regular polygon is inscribed by a circle and the number of sides of the regular polygon is increasing infinite, i.e., when the number of sides is taken very close to infinity, then the regular polygon comes very close to a circle, i.e., almost a circle, so the limiting condition of a regular polygon is a circle.

The limiting conditions of the perimeter and area of a regular polygon are the circumference and area of a circle, respectively. Therefore, limit is related to closeness or approaches.

Example 1

The graph of the function $f(x)$ is given in the figure on the right. Discuss the following questions in a group.



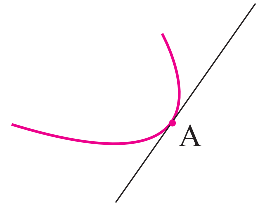
a. How do you define a tangent line?

b. What do you call line segment AB which A and B of the curve passes through point?

c. What changes occur in line AB when point B is closer to point A? Which line does line AB come closer to?

d. When point B comes very close to point A (the distance between the points A and B is almost zero), what will be the limiting value of the line AB in that case?

Solution



a. A straight line passing through only one point of any curve is called a tangent line. The straight line given in the figure is a tangent line at point A.

b. In the above figure, line AB is called a secant line. Lines passing through any two points of any curve are called secant lines.

c. When point B is brought close to point A, line AB comes very close to the tangent line passing through point A.

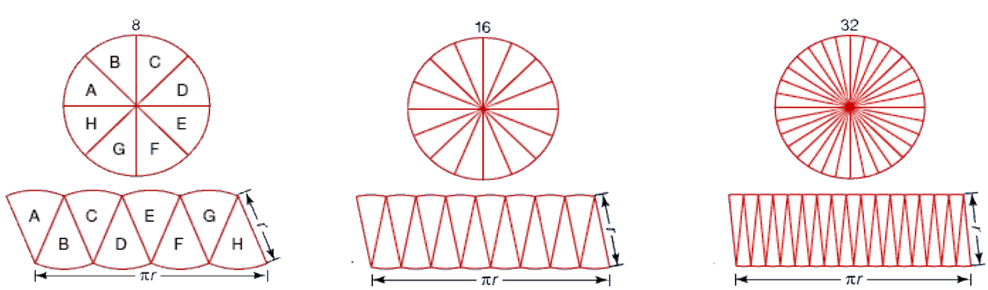
d. When point B comes very close to point A but does not reach point A exactly, in that case line AB is very close to the tangent line at point A, so the secant line AB is the limiting tangent at point A.

When point B comes very close to point A, the secant line AB comes very close to the tangent line and does not cross it. Therefore, the limit of secant is the tangent. In the notation

> When B → A, secant line → is the tangent. Here '→' denotes 'very closeness', 'tends to' or 'approaches to'. Therefore, tangent at a point A = $\lim_{B \to A}$ Secant AB.
>
> The limit is related to closeness or approaches.

Example 2

Have you ever wondered how the area of a circle is πr^2? To prove that the area of a circle is πr^2, we need the concept of limiting value. To understand this, study the figure below.



Take a circle and divide it into equal parts, 8, 16, and 32, and arrange them as shown in the figure. By increasing the number of parts of the circle and arranging them as shown in the figure, a rectangular shape is being formed. If the number of parts is increased and arranged as shown in the figure, the area of the circle approaches the area of the rectangle. That is, when the number of parts becomes infinite, then the limiting area of the circle is the area of the rectangle.

When this is written in symbols,

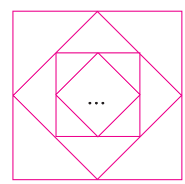
$\lim_{n \to \infty}$ area of circle = The area of the rectangle (n is the number of sector.)

= Length × width

$= \pi r \times r = \pi r^2$

When the number of sectors n. is made infinite and arranged as shown in the figure above, the limiting condition of the circle becomes a rectangle. When one quantity gets very close to an object or number, the object or number to which the other quantity gets very close is the limiting value of the second quantity. The limiting value is related to the proximity or tends to or approaches to of one object to another.

Example 3



16 cm

In the figure on the right, the length of the outer square is 16cm. Then, squares are constructed by adding the midpoints of each square.

a. Write the lengths of the sides of the second, third and fourth squares.
b. How many such squares can be constructed?
c. Estimate the limiting value of the lengths of the sides of the squares.
d. Estimate the limiting value of the perimeter of the squares.
e. Estimate the limiting value of the areas of the squares.

Solution

a. The length of the side of the first square = 16cm
   The length of the side of the second square. $8\sqrt{2}$ cm
   The length of the side of the third square = 8cm
   The length of the side of the fourth square. $4\sqrt{2}$ cm
b. Such squares can be constructed infinitely.
c. While constructing the sequence of the lengths of the sides of the squares, $16, 8\sqrt{2}, 8, 4\sqrt{2}, \ldots$

From the above sequence, if the number of squares is increased, the length of the sides becomes very close to 0. Therefore, the limiting value of the length of the sides of the squares is 0

d. Since the number of squares is infinite, the limit of the length of the side is 0, so the limit of the perimeter of the squares is also 0.

e. Since the number of squares is infinite, the limit of the area of the squares is also 0.

Do the following activities for the limit based on sequences:

Activity 2

Concept of limit based on sequences of numbers

Objectives: To develop the concept of limit based on sequences of numbers.

Problem: Discuss the questions in groups based on the given sequence and draw conclusions:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

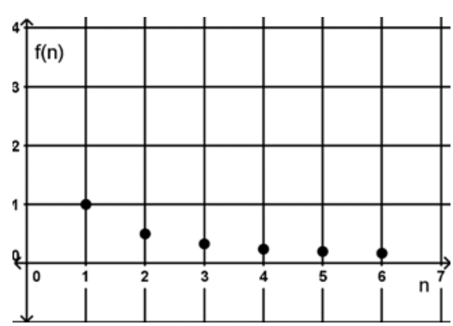
a. Express the formula for the general term of the sequence above as a function.

b. As the number of terms of the sequence increases, which real number does the value of the term approach?

Estimate by completing the table below or looking at the graph.

(n)	{f (n)}
First term (n = 1)	$f(1) = 1$
Second term (n = 2)	$f(2) = \frac{1}{2} = 0.5$
Third term (n = 3)	$f(3) = \frac{1}{3} = \cdots$
Fourth term (n = 4)	$f(4) = \frac{1}{4} = \cdots$
Fifth term (n = 5)	$f(5) = \frac{1}{5} = \cdots$
Tenth term (n = 10)	$f(10) = \frac{1}{10} = \cdots$
Hundred term (n = 100)	$f(100) = \frac{1}{100} = \cdots$
Thousand term (n = 1000)	$f(1000) = \frac{1}{1000} = \cdots$
Ten thousand term (n = 10000)	$f(10000) = \frac{1}{10000} = \cdots$
....	....
Close to ∞	Close to ....

Looking at graph by making ordered pair (n, fn.)



As we increase the number of terms of a sequence and bring it close to infinity, the value of the term of the sequence will get very close to 0. That is, when the value of n is kept close to ∞, the value of fn. will get very close to 0 but not more than 0. In such a case, the limit of the sequence will be 0. That is,

When $n \to \infty$ then $f(n) \to 0$ and $\lim_{n \to \infty} f(n) = 0$

In this way, when the number of terms is kept close to ∞, the real number to which the value of the sequence gets close is the limit of the sequence.

Example 1

Find the limit of the sequence $f(n) = \frac{1}{2^n}$

Solution

We know that, as the value of the term n approaches infinity (∞), the limit of the given sequence is the number to which the value of n approaches infinity (∞). As the value of n approaches infinity,

Number of term **(n)**	The value of that position in the sequence $f(n)$
First term (n = 1)	$f(1) = 1$
Second term (n = 2)	$f(2) = \frac{1}{2^2} = \frac{1}{4} = 0.25$
Third term (n = 3)	$f(3) = \frac{1}{2^3} = \frac{1}{8} = 0.125$
Fourth term (n = 4)	$f(4) = \frac{1}{2^4} = \frac{1}{16} = 0.0625$
Fifth term (n = 5)	$f(5) = \frac{1}{2^5} = \frac{1}{32} = 0.03125$
Tenth term (n = 10)	$f(10) = \frac{1}{2^{10}} = \frac{1}{1024} = 0.00097$
....	....
Clsoe to ∞	Clsoe to ......

According to the above table, when the value of n is very close to ∞, the value of $f(n)$ becomes very close to 0,

So, the limit of the given sequence is 0. In notation, $\lim_{n \to \infty} f(n) = 0$

Example 2

Based on the given sequence, answer the following questions:

3.1, 3.01, 3.001, 3.0001, 3.00001, ...

a. What is the value of the eighth term of the given sequence?

b. As the value of the number of terms increases, to which number will the value of that term in the sequence get closer? Guess.

c. What is the limiting value of the given sequence?

Solution

a.

Number of term (N)	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
The value of that position in the sequence fn.	3.1	3.01	3.001	3.0001	3.00001	3.000001	3.0000001	3.00000001

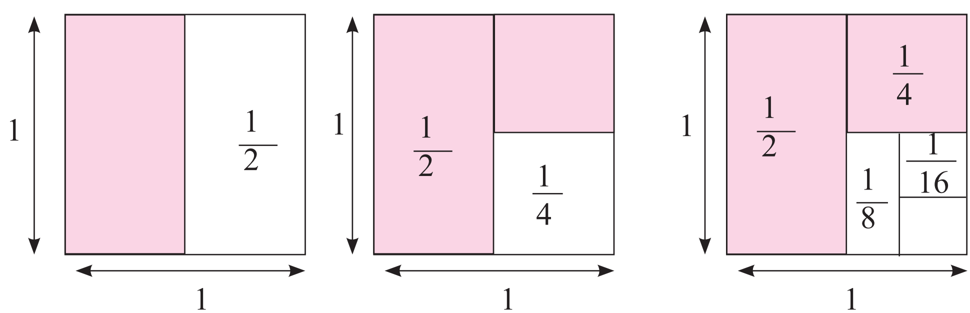
According to the above table, the value of the eighth term of the sequence is 3.00000001.

b. If we increase the value of the number of terms n. and approach infinity the value of the sequence gets very close to 3, i.e., it does not cross the limit of 3.

c. The limit of the given sequence is 3.

Activity 3

A unit square is given in the figure whose area is 1 square unit. In continuous division of the square in half and divide it into two equal rectangles and find the areas of each of the rectangles and their sum.



Each student in the class should solve the above problem and show it to the teacher.

If we divide the square into two rectangles of equal area, the area of each rectangle is $\frac{1}{2}$ square unit. If we divide one of the rectangles in half, the area of one of the square will be $\frac{1}{4}$ square unit. If we divide the square thus formed in half again, the area of one rectangle will be $\frac{1}{8}$ square unit.

And if we continue to divide it in half in this way, will the area of all the rectangles have added up equal to the area of the whole square?

Is, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$ possible?

The area of each of the above rectangles can be written as an infinite series as follows:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

Now, to find the sum of the above series, first of all, we find the partial sum of the series and take a sequence of those partial sums and find the limit of that sequence and the limit of the sequence of partial sums is the sum of the given infinite series.

When finding the partial sum of the given series,

Number of terms of partial sum of the seriesn.	**Interpretation**	**Sum (Sn)**
First Partial Sum (S_1)	Sum of first term of series	$\frac{1}{2}=0.5$
Second Partial Sum (S_2)	Sum of first and second terms of series	$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}=0.75$
Third Partial Sum (S_3)	Sum of first, second and third Terms of series	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}=0.875$
Fourth Partial Sum (S_4)	Sum of first, second, third and fourth terms of series	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}=0.9375$
Fifth Partial Sum S_5	Sum of first to fifth terms of series	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{31}{32}=0.96875$
....	....	....
Let's take n tends to infinity $n \to \infty$		What will be the value of sum of n terms of series ? Guess and discuss.

The sequence of partial sums is as follows,

Sequence = $s_1, s_2, s_3, s_4, s_5, s_6, \ldots$ = 0.5, 0.75, 0.875, 0.9375, 0.96875, …

Observing the sequence of partial sums above, the limit of the sequence is 1.

Therefore, the sum of the given infinite series is 1.

We need limit to find the sum of any infinite series,
The sum of an infinite series is the limit of the sequence of partial sums of that series. If $\sum_{n=1}^{n=\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$ is an infinite series, then its partial sum can be defined as follows:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

....

$s_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$

....

The sequence formed by partial sum is: $s_1, s_2, s_3, s_4, s_5, s_6, \ldots$

The limit of the sequence of partial sums is the sum of the given series.

Example 1

Find the first five partial sums of the infinite series below.

$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$

Solution

The first five partial sums of the above series are as follows:

First, $(s_1) = 4$

Second, $(s_2) = 4 + 2 = 6$

Third, $(s_3) = 4 + 2 + 1 = 7$

Fourth, $(s_4) = 4 + 2 + 1 + \frac{1}{2} = 7.5$

Fifth, $(s_5) = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} = 7.75$

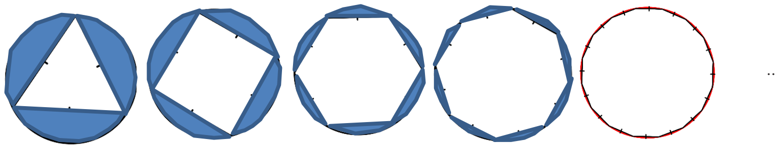
The sequence of partial sums,

$s_1, s_2, s_3, s_4, s_5 = 4, 6, 7, 7.5, 7.75, \ldots$

Thought Provoking Question: Can the sum of all types of infinite series be found? Discuss and draw a conclusion.

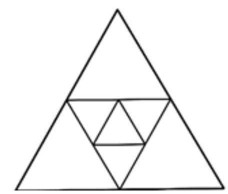
Exercise 6.2

1. Answer the questions by observing the figure below.



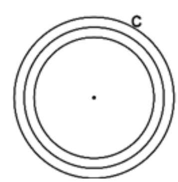
If the number of sides of regular polygons is infinite, what is the limiting value of the area of the shaded part? Estimate.

2. In the figure on the right, a sequence of equilateral triangles is shown. By joining the midpoints of the sides of the previous equilateral triangle, successive equilateral triangles are formed.



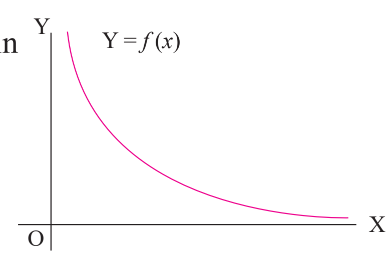
   a. How many such equilateral triangles can be constructed?
   b. If the process of constructing equilateral triangles continues infinitely, what will be the limiting value of the side length of the equilateral triangles? Estimate it.
   c. If the process continues infinitely, what will be the limiting value of the perimeters of the equilateral triangles? Estimate it.
   d. If the process continues infinitely, what will be the limiting value of the areas of the equilateral triangles? Estimate it.

3. In the figure on the right, several concentric circles are drawn inside a circle.



   a. Draw two more such similar figures.
   b. How many concentric circles can be drawn in this way?
   c. If the circles are drawn infinitely in this manner, estimate the limiting value of the circumferences of the circles.
   d. If the circles are drawn infinitely in this manner, what will be the limiting value of the areas of the circles? Estimate it.

4. The graph of the function y = $f(x)$ is presented in the figure on the right. How does y change as the value of x increases? For what value of x, is the value of y equal to 0 ?



5. If n is any natural number, find the limit of the following sequences.

a. $f(n) = \frac{1}{10^n}$ b. $f(n) = \frac{n}{n^2+2}$ c. $f(n) = \frac{1}{n+2}$

d. $f(n) = \frac{1}{2n}$ e. $f(n) = n^2$ f. $f(n) = 2n$

6. Find the limit of the following sequences.

a. 0.1, 0.01, 0.001, 0.0001, ...

b. 7.1, 7.01, 7.001, 7.0001, ...

c. 2.9, 2.99, 2.999, 2.9999, …

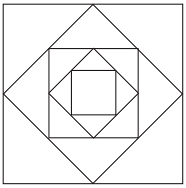
7 Where have you seen patterns of polygons made inside a circle? Collect such examples.

a. What kinds of shapes are found in those patterns?

b. What will be the limiting figure (boundary shape) of those polygons?

8. Take a line segment AB of 10 cm. Divide the line segment into half and find the length of the half. Then, divide that half into half and find its length again. Continue this process of halving and record the lengths obtained to form a sequence. What will be the limiting value of that sequence? Find it.

9 In the figure, the outer square has an area of 16 square units.



By joining the midpoints of the outer square, an inner square is formed.

This process of forming inner squares is shown as a sequence.

How many inner squares can be constructed in this way?

a. Find the side length of each square.

b. Write the sequence of areas of these squares and find its limiting value.

c. Write the sequence of perimeters of these squares and find its limiting value.

d. Find the sum of the areas of all the squares formed in this process.

Answer

1. 0 2. a. infinite b. 0 c. 0 d. 0 3. b. infinite

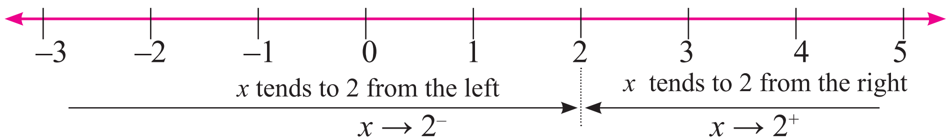
c. 0 d. 0 4. Show to the teacher. 5.a. 0 b. 0 c. 0 d. 0 e. ∞

f. ∞ 6.a. 0 b. 7 c. 3 7 - 9. Show to the teacher.

6.3 Limit of Algebraic Function

Meaning of $x \to a$. To find the limit of an algebraic function, it is very important for us to understand the meaning of $x \to a$.

For that, we need to understand the real number line. We have already studied the set of real numbers and the number line in previous classes. For example, $x \to 2$ means the real numbers whose value of x is very close to 2 but not exactly 2. If we look at this on the real number line



When taking the value of x very close to 2, it should be taken from both the left and the right sides of 2.

When x is taken very close to 2 from the left side,

We have: $x = 1.5, 1.9, 1.99, 1.999, 1.9999, 1.99999, \ldots$

In symbols, $x \to 2^- = 1.5, 1.9, 1.99, 1.999, 1.9999, 1.99999, \ldots$

When x is taken from the right side of 2,

We have: $x = 2.5, 2.1, 2.01, 2.001, 2.0001, 2.00001, \ldots$

In symbols, $x \to 2^+ = 2.5, 2.1, 2.01, 2.001, 2.0001, 2.00001, \ldots$

$\therefore$ Saying $x \to 2$ means that the value of x is taken very close to 2 from both the left and right sides, but x is not exactly equal to 2

Thus, $x \to 2$ represents both $x \to 2^-$ and $x \to 2^+$

In general, $x \to a$ means the value of x is very close to a real number a, but x is not exactly equal to a. When taking values very close to a, they must be taken from both the left and the right sides of a.

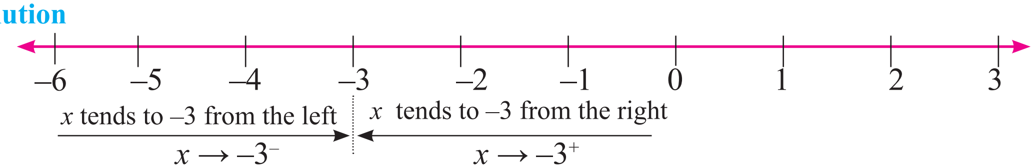
The real numbers that are very close to a from the left side are represented by $x \to a^-$, and those very close to a from the right side are represented by $x \to a^+$.

Hence, $x \to a$ represents both $x \to a^-$ and $x \to a^+$.

Example 1

Write the meaning of $x \to -3$

Solution



$x \to -3$ means that the value of x is very close to –3 from both the right and left sides, but not exactly equal to –3.

The values of x that are very close to –3 from the right side are denoted by $x \to -3^+$, and they are: $x \to -3^+ = -2.9, -2.99, -2.999, -2.9999, \ldots$

The values of x that are very close to –3 from the left side are denoted by $x \to -3^-$, and they are: $x \to -3^- = -3.01, -3.001, -3.0001, -3.00001, \ldots$

Activity 1

Study the following facts and draw conclusions:

First, we know that when any number is divided by itself, the quotient is 1.

For example: $\frac{3}{3}=1, \frac{2}{2}=1, \frac{1}{1}=1$

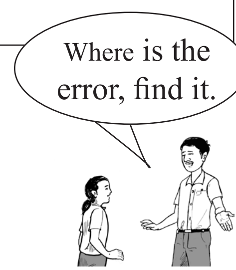
Therefore, $\frac{0}{0}=1$ should be true.

Second, when 0 is divided by any number, the quotient is 0.

For example: $\frac{0}{3}=0, \frac{0}{2}=0, \frac{0}{1}=0$

ence, $\frac{0}{0}=0$ should be true.

$4+5=5+4$
$4-4=5-5$
$\frac{4-4}{5-5}=\frac{5-5}{5-5}$ (Dividing by 5 – 5 on both sides.)
$\frac{0}{0}=1$



Now, from above example, what about $\frac{0}{0}$?

- What value should we take for this?

Can the value of $\frac{0}{0}$ be determined?

Third, according to the principle of division,

if $\frac{a}{b}=c$ then $a=b\times c$, where the value of c is only one.

Similarly, if $\frac{0}{0}=m$, $0=0\times m$, then what value of m will make $0=0\times m$ come true?

For the equation $0=m\times 0$ is true for every value of m, because any number multiplied by zero gives zero. So, $0=m\times 0$ is true for any real value of m.

$m=1$, $0=0\times 1$, True

$m=2$, $0=0\times 2$, True

$m=3$, $0=0\times 3$, True

Therefore, the value of m is not unique. Now, where is the mistake in assuming that $\frac{0}{0}$ has a definite value? From the above example, we can conclude that, the value of m is not qunique.

Since m can be any real number, its value is indeterminate.

Therefore, the value of $\frac{0}{0}$ cannot be determined.

Hence, we say that the expression $\frac{0}{0}$ is an indeterminate form.

The expression $\frac{0}{0}$ is called an indeterminate form because its value cannot be uniquely or certainly determined. Other examples of indeterminate forms include: $\frac{\infty}{\infty}$, 0^0, ∞^0, $\infty-\infty$, 1^∞, $0\times\infty$.

Example 2

In the function $f(x)=\frac{x^2-4}{x-2}$ put the value of $x=0, 1, 2$ and 3. At which value of x in the region is the function in indeterminate form? Find.

Solution

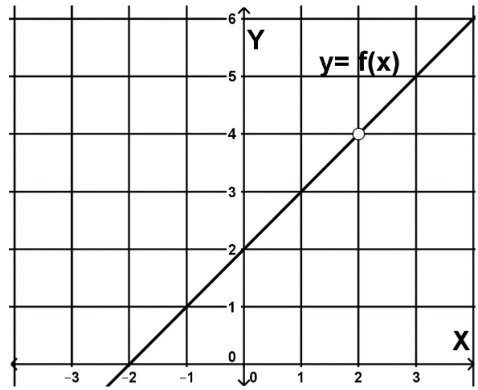
In the given function, $f(x) = \frac{x^2 - 4}{x-2}$

When, $x = 0$ $\quad f(0) = \frac{0^2 - 4}{0-2} = \frac{-4}{-2} = 2$

When, $x = 1$ $\quad f(1) = \frac{1^2 - 4}{1-2} = \frac{-3}{-1} = 3$

When, $x = 2$ $\quad f(2) = \frac{2^2 - 4}{2-2} = \frac{0}{0} =$ indeterminate

When, $x = 3$ $\quad f(3) = \frac{3^2 - 4}{3-2} = \frac{9-4}{1} = 5$



When the value of $x = 2$, then the function is indeterminate form.

Example 3

Based on the above activity, fill in the table below for the function $f(x) = 5x - 3$.

a.

x	–2.1	–2.01	–2.001	–2.0001	–2.00001	–2.000001
$f(x)$	…	…	…	…	…	

b.

x	–1.9	–1.99	–1.999	–1.9999	–1.99999	–1.999999
$f(x)$	…	…	…	…	…	…

Solution

a.

x	–2.1	–2.01	–2.001	–2.0001	–2.00001	–2.000001
$f(x)$	–13.5	–13.05	–13.005	–13.0005	–13.00005	–13.000005

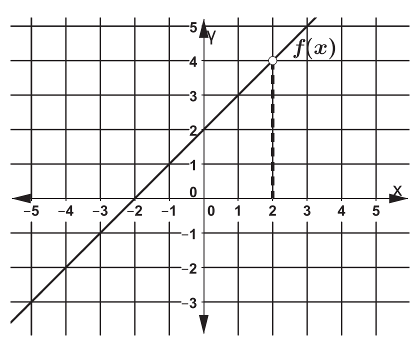
a.

x	–1.9	–1.99	–1.999	–1.9999	–1.99999	–1.999999
f(x)	–12.5	–12.95	–12.995	–12.9995	–12.99995	–12.999995

Example 4

A function $f(x) = \frac{x^2 - 4}{x - 2}$ is given. Its graph is given on the right. Discuss the following questions.

Graph



a. What is the value of $f(x)$ when $x = 2$ as shown in the figure.

b. Study the table below and estimate the value or behaviour of the function at that time when the value of x is close to 2. Or, to which real number is the value of $f(x)$ approaching? Estimate.

a. When the value of x is taken very close to 2 from the left side $(x \to 2^-)$

x	1.9	1.99	1.999	1.9999	…	2
$f(x)$	3.9	3.99	3.999	3.9999	…	?

b. When the value of x is taken very close to 2 from the right side $(x \to 2^+)$

x	2.1	2.01	2.001	2.0001	…	2
$f(x)$	4.1	4.01	4.001	4.0001	…	?

c. What will be the left–hand limit of the function $f(x)$? Estimate it and write it in notation.

d. What will be the right–hand limit of the function $f(x)$? Estimate it and write it in notation.

e. What will be the limit of the function $f(x)$? Estimate it and write it in notation.

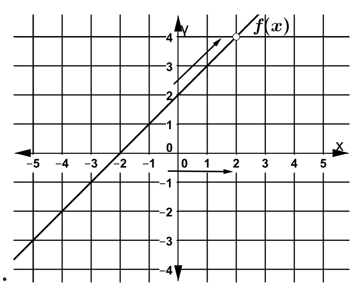
Solution

a. When $x = 2$, $f(2) = \frac{x^2 - 4}{x - 2} = \frac{0}{0}$ = becomes an indeterminate form, which can be clearly understood from the above graph.

b. In fact, since the function becomes indeterminate when $x = 2$, it is necessary to know how the function behaves when x is very close to 2.

Therefore, in table (i), when the value of x is taken very close to 2 from the left side, the value of the function is seen approaching 4.

Graph



Similarly, in table (ii), when the value of x is taken very close to 2 from the right side, the function value is seen approaching 4.

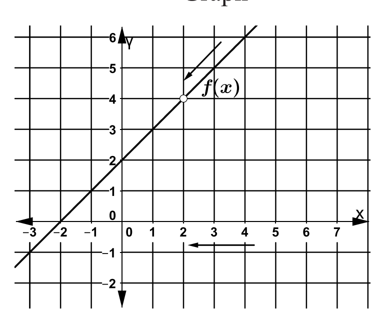
c. When the value of x is taken very close to 2 from the left side, the function value gets very close to 4 and never exceeds 4.

Therefore, the left–hand limit of the function is 4.

In notation, it is written as $x \to 2^-$, $f(x) \to 4$

or, $\lim_{x \to 2^-} f(x) = 4$

Graph



d. When the value of x is taken very close to 2 from the right side, the function value gets very close to 4 and never exceeds 4.

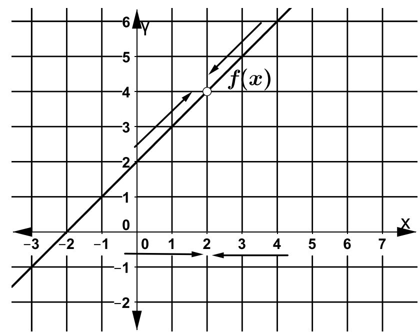
Therefore, the right–hand limit of the function is 4.

In notation: it is written as $x \to 2^+$, $f(x) \to 4$

Or, $\lim_{x \to 2^+} f(x) = 4$

e. Since the right-hand and left-hand limits of the function are equal and both are 4, we can say that when the value of x is taken very close to 2, the value of the function $f(x)$ becomes very close to 4 and never exceeds 4.

Graph



Therefore, when x is taken very close to 2, the limit of the function $f(x)$ is 4.

In notation: it is written as $x \to 2, f(x) \to 4$

That is, $\lim_{x \to 2} f(x) = 4$

Here, taking 2 as a and 4 as 'l', for the limit of the function $f(x)$ to be $l = 4$, the value of x must approach the real number $x = 2$ very closely (from both the right-hand and left-hand sides), and the value of the algebraic function $f(x)$ must approach the real number $l = 4$.

The limit of a function means the behaviour of the function; i.e., we observe where the function approaches when x becomes extremely close to a real number a (from both the right and the left sides). Thus, if the value of x becomes very close to some real number a (from both sides), and the algebraic function $f(x)$ approaches the real number l, then l is called the limit of $f(x)$.

In notation: $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$ and $\lim_{x \to a} f(x) = l$

Limit of a function exists if the right-hand and left-hand limits are equal, if the right-hand and left-hand limits are not equal, then the limit of the function does not exist.

Example 5

Find the limit of the function $f(x) = 4x - 1$ as $x \to -1$.

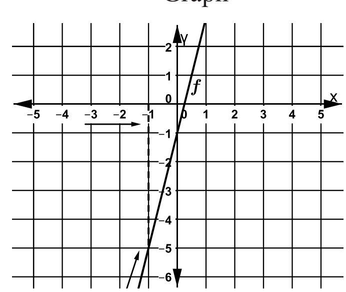
Solution

a. Left–hand limit

By taking values of x from the left side of -1 and constructing a table,

x	–1.1	–1.01	–1.001	–1.0001	–1.00001	…	–1
$f(x)$	–5.4	–5.04	–5.004	–5.0004	–5.00004	…	–5

Graph



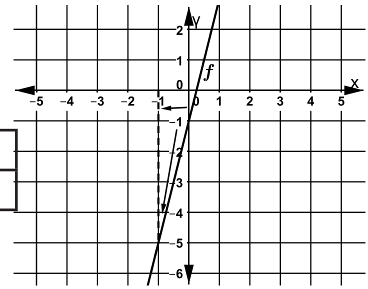
From the above table, when the value of x is taken very close to -1 from the left side, the value of the function $f(x)$ becomes very close to -5.

Therefore, the left–hand limit of the function is -5.

In notation, as $x \to -1^-$, $f(x) \to -5$. That is, $\lim_{x \to -1^-} f(x) = -5$

b. Right–hand limit

Graph



By taking values of x from the right side of –1 and constructing a table,

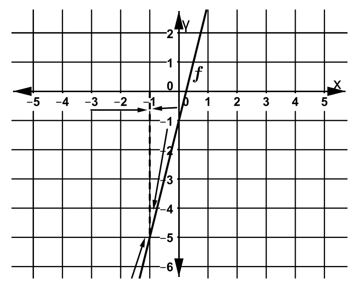
x	–0.9	–0.99	–0.999	–0.99999	–0.99999	…	–1
$f(x)$	–4.6	–4.96	–4.996	–4.9996	–4.99996	…	–5

From the table, when the value of x approaches –1 from the right side, the value of the function $f(x)$ becomes very close to –5.

Therefore, the right–hand limit of the function is –5.

In notation, as $x \to -1^+$, $f(x) \to -5$. That is, $\lim_{x \to -1^+} f(x) = -5$

Graph



c. Limit of the function

Since the right-hand and left-hand limits are equal, the limit of the function as x approaches –1 is –5.

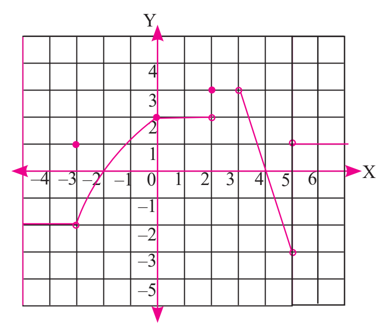
In notation, as $x \to -1$, $f(x) \to -5$.

That is, $\lim_{x \to -1} f(x) = -5$

Example 6

The graph of $y = f(x)$ is given on the right side.

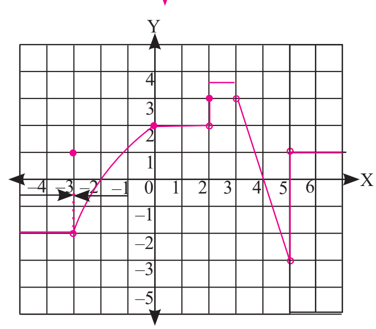
By observing the graph, state whether the limit exists at the given points or not, with reasons.



a. $\lim_{x \to -3} f(x)$ b. $\lim_{x \to 5} f(x)$

c. $\lim_{x \to 2} f(x)$

Solution



a. In the given graph, when the value of x approaches –3 from the right side, the value of the function becomes very close to –2.

That is, as $x \to -3^+$, $f(x) \to -2$

Therefore, the right–hand limit of the function is -2.

Similarly, when the value of x approaches –3 from the left side, the value of the function becomes very close to –2.

That is, as $x \to -3^-$, $f(x) \to -2$

Therefore, the left–hand limit of the function is –2

Since the right-hand and left-hand limits of the function are equal, the limit of the function is –2.

i.e., $\lim_{x \to -3} f(x) = -2$

b. In the given graph, when the value of x approaches 5 from the right side, the value of the function becomes very close to 1.

i.e., as $x \to 5^+, f(x) \to 1$

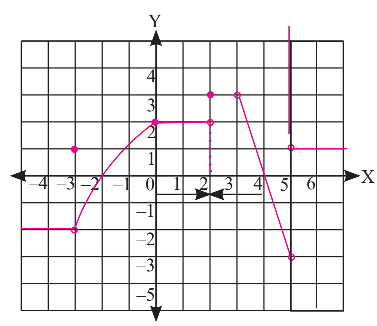
Similarly, when the value of x approaches 5 from the left side, the value of the function becomes very close to –3.

i.e., as $x \to 5^-$, $f(x) \to -3$ Therefore, the left-hand limit of the function is –3.

Since the right-hand and left-hand limits are not equal, the limit of the function cannot be determined.

i.e., $\lim_{x \to 5} f(x)$ = does not exist

c. In the given graph, when the value of x approaches 2 from the right side, the value of the function can not be determined; therefore, the right-hand limit does not exist and becomes infinite (∞).



Similarly, when the value of x approaches 2 from the left side, the value of the function becomes very close to 2.

That is, as $x \to 2^-, f(x) \to 2$

Therefore, the left-hand limit of the function is 2.

Since the right-hand and left-hand limits are not equal, the limit of the function cannot be determined (the limit does not exist).

That is, $\lim_{x \to 2} f(x)$ = does not exist

Exercise 6.3

1. a. Write the meaning of $x \to -3$ b. Write the meaning of $x \to 5^-$.

   c. Write the meaning of $\lim_{x \to 2} f(x) = -5$ d. Write the meaning of $\lim_{x \to -2^-} f(x) = 1$.

   e. Write the meaning of $\lim_{x \to -3^+} f(x) = -4$.

   f. When the value of x is very close to m (from both the right and left sides), which of the following notations correctly represent the limit of $f(x)$?

   i. $\lim_{x \to m^+} f(x)$ ii. $\lim_{x \to m^-} f(x)$ iii. $\lim_{x \to -m^+} f(x)$ iv. $\lim_{x \to m} f(x)$

2. A function is given: $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

   a. Find the value of $f(x)$ by substitute $x = 0, 1, 4$ and 9

   b. For what value of x does the function become indeterminate? Write it.

3. A function is given: $f(x) = \dfrac{x^2-16}{x-4}$

   a. Find the value of $f(x)$ by substituting $x = 0, 1, 3$ and 4.

   b. For what value of x does the function become indeterminate write it.

4. Complete the following table and based on the completed table, find the limit of the given function.

   a. $\lim\limits_{x \to 3} (x+2)$

x	2.9	2.99	2.999	2.9999	...	3
$x+2$					...	?

x	3.1	3.01	3.001	3.0001	...	3
$x+2$					...	?

   b. $\lim\limits_{x \to -2} (2x-3)$

x	–2.1	–2.01	–2.001	–2.0001	...	–2
$2x-3$					...	?

x	–1.9	–1.99	–1.999	–1.9999	...	–2
$2x-3$					‘	?

   c. $\lim\limits_{x \to 4} (x^2+2)$

x	3.9	3.99	3.999	3.9999	...	4
x^2+2					...	?

x	4.1	4.01	4.001	4.0001	...	4
x^2+2					...	?

   d. $\lim\limits_{x \to -3} \left(\dfrac{x^2-9}{x+3}\right)$

x	–3.1	–3.01	–3.001	–3.0001	...	–3
$\dfrac{x^2-9}{x+3}$					...	?

x	–2.9	–2.99	–2.999	–2.9999	...	–3
$\dfrac{x^2-9}{x+3}$					...	?

5. a. A function is given: $f(x) = \frac{x^2 - 4}{x - 2}$

   i. What is the value of f(2)?

   ii. Find the values of $f(1.9)$, $f(1.99)$, $f(1.999)$, $f(1.9999)$ and determine the left-hand limit of the function as x approaches 2. Write it in notation.

   iii. Find the values of $f(2.01)$, $f(2.001)$, $f(2.0001)$, $f(2.00001)$ and determine the right-hand limit of the function as x approaches 2. Write it in notation.

   iv. What will be the limiting value of the function? Write it in notation.

   b. A function is given: $f(x) = \frac{x^2 - 9}{x - 3}$

   i. What is the value of f(3)?

   ii. Find the values of $f(2.9)$, $f(2.99)$, $f(2.999)$, $f(2.9999)$ and deter mine the left-hand limit of the function as x approaches 3. Write it in notation.

   iii. Find the values of $f(3.01)$, $f(3.001)$, $f(3.0001)$, $f(3.00001)$ and determine the right-hand limit of the function as x approaches 3 . Write it in notation.

   iv. What will be the limiting value of the function? Write it in a notation.

6. Construct tables to determine the limit at the specified point.

   a. $\lim_{x \to 2} (3x+1)$ b. $\lim_{x \to -3} (2x-3)$ c. $\lim_{x \to -2} (4x)$

   d. $\lim_{x \to 1} (x^2 - 2)$ e. $\lim_{x \to -2} (3x^2 + 1)$ f. $\lim_{x \to -2} (2x^2 - 3)$

   g. $\lim_{x \to 3} \left(\frac{x - 1}{x + 2}\right)$ h. $\lim_{x \to -1} \left(\frac{-2}{x + 1}\right)$ i. $\lim_{x \to -3} \left(\frac{x - 1}{x + 2}\right)$

   j. $\lim_{x \to 2} \left(\frac{2}{x - 2}\right)$ k. $\lim_{x \to 3} \left(\frac{2x - 6}{x - 3}\right)$ l. $\lim_{x \to 2} \left(\frac{1}{4x - 8}\right)$

7. Construct tables to determine the limit at the point specified point.

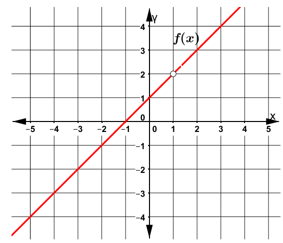
   a. $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3}\right)$ b. $\lim_{x \to 9} \left(\frac{x - 9}{x - 3}\right)$ c. $\lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4}\right)$

   d. $\lim_{x \to 2} \left(\frac{x^2 - 4}{x^2 + x - 6}\right)$ e. $\lim_{x \to -3} \left(\frac{x^2 + 6x + 6}{x^2 - 9}\right)$ f. $\lim_{x \to 1} \left(\frac{x^2 + 1}{x^2 - 1}\right)$

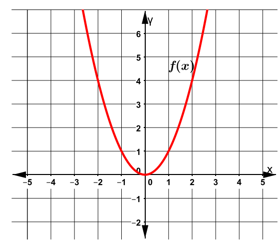
8. Prove that:

a. $\lim_{x \to 1}\left(\frac{x^3-1}{x-1}\right) = 3$ b. $\lim_{x \to 3}\left(\frac{x^2-2x+4}{x^2+1}\right) = \frac{5}{2}$ c. $\lim_{x \to 2}\left(\frac{x^3-8}{x-2}\right) = 12$

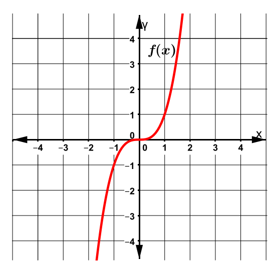
9. Study the graph and answer the following questions:



a. If $f(x) = \frac{x^2-1}{x-1}$, find the limit of $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1} f(x)$.

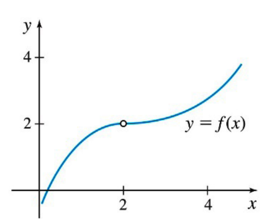


b. If $f(x) = x^2$, find the limit of $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$

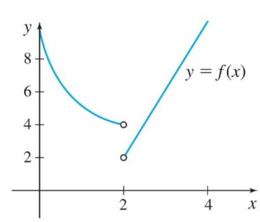


c. If $f(x) = x^3$, find the limit of $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1} f(x)$.

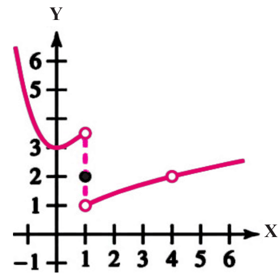
d. From the graph y= $f(x)$ on the right, find the limit of $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2} f(x)$.



10. From the graph on the right, state whether the limit of $f(x)$ at point 2 can be determined or not. Write the reason clearly.



11. Look at the graph on the right, state whether the limit of $f(x)$ can be determined or not at $x = 1$. Write the reason clearly.



Answer

1. Show to the teacher. 2. a. 3, 4, 5, $\frac{0}{0}$ b. $x = 9$ 3. a. 4, 5, 7, $\frac{0}{0}$

b. $x = 4$ 4. a. 5 b. –7 c. 18 d. –6

5. a. i. $\frac{0}{0}$ ii. $\lim_{x \to 2^-} f(x) = 4$ iii. $\lim_{x \to 2^+} f(x) = 4$ iv. $\lim_{x \to 2} f(x) = 4$

b. i. $\frac{0}{0}$ ii. $\lim_{x \to 3^-} f(x) = 6$ iii. $\lim_{x \to 3^+} f(x) = 6$ iv. $\lim_{x \to 3} f(x) = 6$

6. a. 7 b. –9 c. –8 d. –1 e.13 f. 5 g. $\frac{2}{5}$

h. does not exist i. 4 j. does not exist k. 2 l. does not exist

7. a. 6 b. 6 c. 8 d. $\frac{4}{5}$ e. 0 f. does not exist

9. a. 2, 2, 2 b. 1, 1, 1 c. 1, 1, 1 d. 2, 2, 2

10. does not exist 11. does not exist